Chapter 6 HW Solution

Starr Problem. The figure below shows the RR manipulator with attached link frames. There are also frames \{C_1\} and \{C_2\} at the center of the links (with the same orientation as frames \{1\} and \{2\}).

![Robot Diagram](image_url)

The table of link parameters for this robot is

<table>
<thead>
<tr>
<th>i</th>
<th>α_{i-1}</th>
<th>a_{i-1}</th>
<th>d_i</th>
<th>θ_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>θ_1</td>
</tr>
<tr>
<td>2</td>
<td>-90°</td>
<td>l_1</td>
<td>0</td>
<td>θ_2</td>
</tr>
</tbody>
</table>

and the two transformation matrices are

\[ ^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \end{bmatrix}. \tag{1} \]

The inertia matrices for both links have the same structure (since in each case the y and z axes are normal to the link):

\[ I_C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{2} \]

where in each case \( I = m l^2 / 12 \).

I wrote a MATLAB script (which used the Symbolic Math Toolbox) to perform the outward kinematic iterations and inward force/moment iterations (6.45–6.53), and finally obtained the following matrices:

\[ M = \begin{bmatrix} \frac{1}{3} m_1 l_1^2 + \frac{1}{3} m_2 l_2^2 c_2^2 + m_2 (l_1^2 + l_1 l_2 c_2) & 0 \\ 0 & \frac{1}{3} m_2 l_2^2 \end{bmatrix}, \tag{3} \]

\[ V = \begin{bmatrix} (-m_2 l_1 l_2 s_2 - \frac{2}{3} m_2 l_2^2 s_2 c_2) \dot{θ}_1 \dot{θ}_2 \\ \frac{1}{3} m_2 l_2^2 s_2 c_2 + \frac{2}{3} m_2 l_1 l_2 s_2 \dot{θ}_1^2 \\ \frac{1}{2} (m_1 + m_2) g l_1 c_1 + \frac{1}{2} m_2 g l_2 c_1 c_2 \\ -\frac{1}{2} m_2 g l_2 s_1 s_2 \end{bmatrix}, \tag{4} \]

\[ G = \begin{bmatrix} \frac{1}{2} (m_1 + m_2) g l_1 c_1 + \frac{1}{2} m_2 g l_2 c_1 c_2 \\ -\frac{1}{2} m_2 g l_2 s_1 s_2 \end{bmatrix}. \tag{5} \]

Note that mass matrix \( M \) is diagonal, indicating that there is no dynamic coupling. This means the inertia torque from one link does not affect the other links. There is coupling in other ways, however. Compare this \( M \) with the one from the Programming Exercises, where there is dynamic coupling.
6.11. Since the $\hat{Z}_1$ axis is the axis of rotation, the moment of inertia $I_{zz1}$ is the inertia that is being rotated. From your control systems class, you learned that a gear train reduces the inertia “felt” by the motor by a factor of $n^2$, where $n$ is the gear ratio ($n > 1$ for a speed reducing gear train). Hence the total inertia which the motor must accelerate is:

$$I_{\text{total}} = I_m + \frac{I_{zz1}}{n^2} = I_m + \frac{I_{zz1}}{100^2} = I_m + \frac{I_{zz1}}{10,000}$$

(6)

Thus the motor’s own inertia may be dominant! (Gear ratios of $\approx 100$ are not uncommon)

Problem 6.12. Refer to the figures below; frame $\{1\}$ and joint angle $\theta_1(t)$ are shown. Since $^1PC = [2 \ 0 \ 0]^T$, we know that $x_2$ must pass through the center of mass.

The rotation matrix $^0\mathbf{R}$ is needed to find the reaction force on the support; this is a standard $\mathbf{R}_Z(\theta)$, given by

$$^0\mathbf{R} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(7)

The inertia matrix about the center of mass for the link can be expressed symbolically in frame $\{C\}$ as

$$^C\mathbf{I}_1 = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

(8)

Note that since $^C\mathbf{I}_1$ is diagonal then the axes of frame $\{1\}$ must constitute principal axes for link 1. Since this implies that the $xy$, $xz$, and $yz$ planes form planes of symmetry for the body, the poor drawing of the original figure is even more apparent! Oh, well...this is still a pretty good textbook.

(a) Find angular acceleration of link $^1\dot{\omega}_1$ and acceleration of center of mass $^1\dot{v}_{C1}$.

Outward propagations: $i = 0 \rightarrow 1$. From inspection we can see that

$$^0\mathbf{\omega}_0 = [0 \ 0 \ 0]^T, \ ^0\mathbf{\omega}_0 = [0 \ 0 \ 0]^T, \ ^0\mathbf{v}_0 = [0 \ 0 \ 0]^T, \ ^0\mathbf{\dot{v}}_0 = [0 \ 0 \ g]^T$$

(9)

Note that the gravitation acceleration “$g$” is for purposes of modeling gravity and not really a kinematic acceleration. Using the propagation equations (6.45)-(6.50), I found that

$$^1\omega_1 = \begin{bmatrix} 0 \\ \dot{\theta}_1 \end{bmatrix}, \ ^1\dot{\omega}_1 = \begin{bmatrix} 0 \\ \dot{\theta}_1 \end{bmatrix}, \ ^1\dot{v}_1 = \frac{1}{\dot{\theta}_1}^0\mathbf{R}^0\mathbf{\dot{v}}_0 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

(10)

$$^1\dot{v}_{C1} = ^1\dot{\omega}_1 \times ^1\mathbf{P}_C + ^1\omega_1 \times (^1\omega_1 \times ^1\mathbf{P}_C) + ^1\dot{v}_1 = \begin{bmatrix} -l\dot{\theta}_1^2 \\ l\dot{\theta}_1 \\ g \end{bmatrix}$$

(11)
(b) Find the required actuator torque $\tau_1(t)$.

The resultant force and moment as

$$\begin{align*}
1F_1 &= m^1\dot{v}_C = \begin{bmatrix}
-ml\dot{\theta}_1^2 \\
ml\dot{\theta}_1 \\
mg
\end{bmatrix}, \\
1N_1 &= C^1I_1^1\dot{\omega}_1 + \dot{\omega}_1 \times (1^1\omega_1 \times 1^1P_C) = \begin{bmatrix}
0 \\
0 \\
I_{zz}\dot{\theta}_1
\end{bmatrix}
\end{align*}$$  \hfill (12)

**Inward propagations:** The results I obtained are

$$\begin{align*}
1f_1 &= 1F_1, \\
1n_1 &= 1F_1 \times 1^1F_1 = \begin{bmatrix}
0 \\
-mgl \\
(I_{zz} + ml^2)\dot{\theta}_1
\end{bmatrix}
\end{align*}$$  \hfill (13)

At this point we can find joint torque $\tau_1$ as the $Z$ component of $1n_1$; this is

$$\tau_1 = (I_{zz} + ml^2)\dot{\theta}_1$$  \hfill (14)

(c) Reaction force and moment on the support. The reaction force and moment on the support are:

$$\begin{align*}
-1f_1 &= \text{force exerted on link 0 (support) by link 1} \\
-1n_1 &= \text{moment exerted on link 0 (support) by link 1}
\end{align*}$$

However, we would like to express these in frame $\{0\}$. This can be done with the $R$ matrix,

$$\begin{align*}
-0f_1 &= -0^1R^1f_1 = - \begin{bmatrix}
c_1 & -s_1 & 0 \\
s_1 & c_1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
-ml\dot{\theta}_1^2 \\
ml\dot{\theta}_1 \\
mg
\end{bmatrix} = \begin{bmatrix}
ml\dot{\theta}_1^2c_1 + ml\dot{\theta}_1s_1 \\
ml\dot{\theta}_1^2s_1 - ml\dot{\theta}_1c_1 \\
-mg
\end{bmatrix}
\end{align*}$$  \hfill (15)

$$\begin{align*}
-0n_1 &= -0^1R^1n_1 = - \begin{bmatrix}
c_1 & -s_1 & 0 \\
s_1 & c_1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 \\
-mgl \\
(I_{zz} + ml^2)\dot{\theta}_1
\end{bmatrix} = \begin{bmatrix}
-mgls_1 \\
-mglc_1 \\
-(I_{zz} + ml^2)\dot{\theta}_1
\end{bmatrix}
\end{align*}$$  \hfill (16)

**Problem 6.14.** The incorrect terms in the EOMs for the RP manipulator I found were:

$$\begin{align*}
\tau_1 &= m_1(d_1^2 + d_2^2)\dot{\theta}_1 + m_2d_1^2\dot{\theta}_1 + 2m_2d_2\dot{d}_2\dot{\theta}_1 + g\cos(\theta_1)[m_1(d_1 + d_2\dot{\theta}_1) + m_2(d_2 + d_2\dot{\theta}_1)] \\
f_2 &= \tau_2 = m_1d_2\dot{\theta}_1 + m_2d_2 - m_1d_1d_2 - m_2d_2\dot{\theta}_2^2 + m_2(d_2 + 1)g\sin(\theta_1)
\end{align*}$$

My reasoning was based on each term having the correct units of (M L$^2$)/T$^2$ (i.e. torque) in the first EOM, and units of (M L)/T$^2$ (i.e. force) in the second EOM.