Chapter 5 HW Solution

**Problem 5.1.** If you take the Jacobian in frame \{0\} from (5.67), then set its determinant to zero, you will obtain the same singular positions as in frame \{3\}. This must be true because singularities are a fundamental property of the mechanism.

**Problem 5.3.** In the work below, I'm not going to show all the intermediate expressions; just the propagation results at each stage. If there are questions we can discuss the procedure.

**Velocity Propagation.** Starting at frame \{0\} and working towards frame \{4\}:

\[
\begin{align*}
1\omega &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, & 1v &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, & 2\omega_2 &= \begin{bmatrix} s_2 \dot{\theta}_1 \\ c_2 \dot{\theta}_1 \\ \theta_2 \end{bmatrix}, & 2v_2 &= \begin{bmatrix} 0 \\ 0 \\ -L_1 \dot{\theta}_1 \end{bmatrix} \\
3\omega_3 &= \begin{bmatrix} s_{23} \dot{\theta}_1 \\ c_{23} \dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}, & 3v_3 &= \begin{bmatrix} L_2 s_3 \dot{\theta}_2 \\ L_2 c_3 \dot{\theta}_2 \\ -L_1 \dot{\theta}_1 - L_2 c_2 \dot{\theta}_1 \end{bmatrix}, & 4\omega_4 &= \begin{bmatrix} L_2 s_3 \dot{\theta}_2 \\ (L_2 c_3 + L_3) \dot{\theta}_2 + L_3 \dot{\theta}_3 \\ (L_1 - L_2 c_2 - L_3 c_23) \dot{\theta}_1 \end{bmatrix}
\end{align*}
\]

Since \(4v_4 = 4J\dot{\theta}\), we can identify the Jacobian in \{4\} as

\[
\begin{bmatrix}
0 & L_2 s_3 & 0 \\
0 & L_2 c_3 + L_3 & L_3 \\
-L_1 - L_2 c_2 - L_3 c_23 & 0 & 0
\end{bmatrix} \tag{1}
\]

**Force Propagation.** We apply the force and moment \(4f_4 = [f_x \ f_y \ f_z]^T\), \(4n_4 = [0 \ 0 \ 0]^T\), and propagate from frame \{4\} down to frame \{0\}:

\[
\begin{align*}
3f_3 &= 4f_4 = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}, & 3n_3 &= \begin{bmatrix} 0 \\ -L_3 f_z \\ L_3 f_y \end{bmatrix}, & 2f_2 &= \begin{bmatrix} c_3 f_x - s_3 f_y \\ s_3 f_z + c_4 f_y \\ f_z \end{bmatrix}, & 2n_2 &= \begin{bmatrix} L_3 s_3 f_z \\ (L_2 - L_3 c_3) f_z \\ L_2 s_3 f_x + (L_3 + L_2 c_3) f_y \end{bmatrix} \\
1f_1 &= \begin{bmatrix} c_{23} f_x - s_{23} f_y \\ -f_z \\ s_{23} f_z + c_{23} f_y \end{bmatrix}, & 1n_1 &= \begin{bmatrix} * \\ (-L_1 - L_2 c_2 - L_3 c_23) f_z \end{bmatrix}
\end{align*}
\]

Since the joint torques \(\tau_i\) are the \(Z\)-component of \(i^n n_i\), and since the Jacobian relates joint torques and tip forces by \(\tau = J^T f\), we have

\[
\begin{bmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{bmatrix} = \begin{bmatrix}
0 & 0 & -L_1 - L_2 c_2 - L_3 c_23 \\
L_2 s_3 & L_2 c_3 + L_3 & 0 \\
0 & L_3 & 0
\end{bmatrix} \begin{bmatrix}
f_x \\
f_y \\
f_z
\end{bmatrix} \tag{2}
\]

and the Jacobian is again revealed.

**Differentiation of Forward Kinematics.** When you multiply out all the \(T\) matrices, you get

\[
\begin{align*}
x &= c_1 (L_1 + L_2 c_2 + L_3 c_23) \\
y &= s_1 (L_1 + L_2 c_2 + L_3 c_23) \\
z &= L_2 s_2 + L_3 s_23
\end{align*} \tag{3} \tag{4} \tag{5}
\]
The elements of the Jacobian in \{0\} are

\[
0J = \begin{bmatrix}
\frac{\partial x}{\partial \theta_1} & * & * \\
* & \frac{\partial z}{\partial \theta_3}
\end{bmatrix}
\]

where I’m sure you can figure out the “*” entries. After a fair bit of differentiation, I found the result

\[
0J = \begin{bmatrix}
-(L_1 + L_2 c_2 + L_3 c_{23}) s_1 & -(L_2 s_2 + L_3 s_{23}) c_1 & -L_3 c_1 s_{23} \\
(L_1 + L_2 c_2 + L_3 c_{23}) c_1 & -(L_2 s_2 + L_3 s_{23}) s_1 & -L_3 s_1 s_{23} \\
0 & L_2 c_2 + L_3 c_{23} & L_3 c_{23}
\end{bmatrix}
\]

(6)

I just didn’t have the stomach to multiply \(4_0R\) times \(0J\) to get \(4J\), although it should simplify to the same result as the previous two methods.

**Problem 5.4.** Singularities are revealed by setting \(|J| = 0\), and the determinant of a matrix is unchanged by transposing it; therefore singularities exist in the same configurations in the force domain as in the position domain.

**Problem 5.8.**

The Jacobian of the 2-link RR manipulator (in frame \(\{3\}\)) is

\[
3J = \begin{bmatrix}
l_1 s_2 & 0 \\
l_1 c_2 + l_2 & l_2
\end{bmatrix} = [V_1 \quad V_2]
\]

(7)

For both columns to be orthogonal and of equal magnitude column \(V_1\) must be equal to \([l_2 \quad 0]^T\), thus

\[
l_1 s_2 = l_2 \\
l_1 c_2 + l_2 = 0
\]

(8) \hspace{1cm} (9)

First get rid of \(\theta_2\) by squaring and adding (8) and (9); this will yield the relationship between \(l_1\) and \(l_2\); we get

\[
l_1 + 2l_2 c_2 = 0 \implies c_2 = -\frac{l_1}{2l_2} \text{ (note that } c_2 \text{ is negative)}
\]

(10)

Then substituting the expression for \(c_2\) in (10) into (9) we get

\[
l_1 \left(-\frac{l_1}{2l_2}\right) + l_2 = 0 \implies l_1^2 = 2l_2^2
\]

(11)

so the relationship between the link lengths is

\[
\frac{l_1}{l_2} = \pm \sqrt{2} \implies l_1 = \sqrt{2} l_2
\]

(12)

Note the although we have the mathematically meaningful “±” sign, only the “+” sign is physically meaningful.

Now for angle \(\theta_2\); from the middle term of (10) we see that \(c_2\) is negative. From our work with the \(\text{atan2}(\sin, \cos)\) or \(\text{atan2}(y, x)\) or \(\text{atan2}(\text{Im}, \text{Re})\) function (whichever one is easiest for you to visualize), this places the angle \(\theta_2\) in either quadrant II or III (left half complex plane). The sine of \(\theta_2\) is given by (8), and in accordance with (11),

\[
s_2 = \frac{l_2}{l_1} = \pm \frac{1}{\sqrt{2}}
\]

(13)

and we have (at last!) the solution for \(\theta_2\) as

\[
\theta_2 = \text{atan2}(\pm 0.707, -0.707) = \pm 135^\circ
\]

(14)

I’ll draw you this “isotropic” mechanism during class.
Problem 13. This is a simple application of the Jacobian transpose in relating joint torques and static tip force. It is important that both tip force and Jacobian be expressed in the same frame; that is the case here, so we can simply evaluate:

\[ \tau = 0^T J^T \mathbf{f}. \]  

For the case at hand, we have

\[
\begin{bmatrix}
\tau_1 \\
\tau_2
\end{bmatrix} =
\begin{bmatrix}
-l_1 s_1 - l_2 s_{12} \\
l_1 c_1 + l_2 c_{12}
\end{bmatrix}^T
\begin{bmatrix}
l_1 s_1 - l_2 s_{12} \\
10
\end{bmatrix} =
\begin{bmatrix}
-10l_1 s_1 - 10l_2 s_{12} \\
-10l_2 s_{12}
\end{bmatrix}
\]  

MATLAB Exercise 5. Here we are using the Jacobian inverse for doing Cartesian velocity control (rate control). In a technique called Resolved Motion Rate Control we use \( J^{-1} \) in the control loop to obtain the required joint rates to guide the last link with the desired Cartesian velocity \( \mathbf{v} \).

Given the link lengths, initial position, and desired (constant) velocities, you were asked to plot the following fine quantities vs time:

1. The three joint rates \( \dot{\theta}_i(t) \) (these were found using \( J^{-1} \mathbf{v} \)).
2. The three joint angles \( \theta_i(t) \) (these were found using simple integration \( \theta_{i+1} = \theta_i + \dot{\theta}_i dt \)).
3. The three “user form” Cartesian components \( [x \ y \ \phi] \) (found using forward kinematics).
4. Jacobian determinant (found using MATLAB \texttt{det(J)}).
5. The three joint torques \( \tau_i \) required to equilibrate a given wrench \( \mathbf{W} \) (found using \( J^T \mathbf{W} \)).
6. (not required) I thought it would be interesting to plot the condition number of Jacobian \( \text{J} \) (done using MATLAB \texttt{cond(J)}).

The six plots are shown below. We will run the graphical simulation of the motion during class.

![Joint angles vs time.](a) Joint angles vs time. | ![Joint rates vs time.](b) Joint rates vs time. |  
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Figure 1: Joint angles and rates during the motion.
Figure 2: Cartesian position and joint torque during the motion.

Figure 3: Jacobian determinant and condition number during the motion.