Chapter 4 HW I Solution

Problem 4.1. Let points \( A, B, C, D \) be the four “corners” of the four-bar linkage, as shown in Figure 1. Express the position of either end of link 2 two ways — the two scalar constraint equations will result. For example

\[
\begin{align*}
\mathbf{r}_B &= L_1 c_1 \mathbf{i} + L_1 s_1 \mathbf{j} \\
\mathbf{r}_C &= \mathbf{r}_B + L_2 c_2 \mathbf{i} + L_2 s_2 \mathbf{j} \\
\mathbf{r}_C &= L_4 \mathbf{i} + L_3 c_3 \mathbf{i} + L_3 s_3 \mathbf{j}
\end{align*}
\]

Equate both expressions for \( \mathbf{r}_C \) and resolve into \( i \) and \( j \) components:

\[
\begin{align*}
L_1 c_1 + L_2 c_2 &= L_3 c_3 + L_4 \\
L_1 s_1 + L_2 s_2 &= L_3 s_3
\end{align*}
\]

Thus the two constraint equations are:

\[
\begin{align*}
L_1 c_1 + L_2 c_2 - L_3 c_3 - L_4 &= 0 \\
L_1 s_1 + L_2 s_2 - L_3 s_3 &= 0
\end{align*}
\]

Problem 4.7. The (fixed) \( Z \) axis is given (consider it part of fixed frame \( A \)). I attached a moving frame \( B \) to the arm, as shown in Figure 2. Use the relative velocity equation, with everything expressed in frame \( B \) (easier). We have

\[
\mathbf{v}_P = \mathbf{v}_B + \mathbf{\omega}_B \times \mathbf{r}_P + \mathbf{\omega}_B \times \mathbf{v}_P
\]

where the terms in (8) are

\[
\begin{align*}
\mathbf{v}_B &= \dot{\phi} \mathbf{b}_2 \times L_1 \mathbf{b}_1 = -L_1 \dot{\phi} \mathbf{b}_3 \\
\mathbf{\omega}_B \times \mathbf{r}_P &= \dot{\phi} \mathbf{b}_2 \times L_2 (s\theta \mathbf{b}_1 - c\theta \mathbf{b}_2) = -L_2 \dot{\phi} s\theta \mathbf{b}_3 \\
\mathbf{\omega}_B \times \mathbf{v}_P &= \dot{\theta} \mathbf{b}_3 \times (s\theta \mathbf{b}_1 - c\theta \mathbf{b}_2) = L_2 \dot{\theta} c\theta \mathbf{b}_1 + L_2 \dot{\theta} s\theta \mathbf{b}_2
\end{align*}
\]

Thus the velocity of point \( P \) is

\[
\mathbf{v}_P = L_2 \dot{\theta} c\theta \mathbf{b}_1 + L_2 \dot{\theta} s\theta \mathbf{b}_2 - (L_1 \dot{\phi} + L_2 \dot{\phi} s\theta \mathbf{b}_3)
\]

The virtual displacement comes directly from the velocity:

\[
\mathbf{\delta r}_P = L_2 \dot{\theta} c\theta \mathbf{b}_1 + L_2 \dot{\theta} s\theta \mathbf{b}_2 - (L_1 + L_2 s\theta) \dot{\phi} \mathbf{b}_3
\]

Problem 4.8. Attach a fixed frame \( A \) at point \( O \), aligned as shown. Let \( \phi \) be the angle of link 2 from horizontal.
(a) Velocity Method. Using the “two points on a rigid body” method, the velocity of point \( P \) is

\[
\mathbf{v}_P = \mathbf{v}_B + \omega_2 \times \mathbf{r}_{P/B}
\]

where

\[
\mathbf{v}_B = \omega_1 \times \mathbf{r}_{B/O} = \dot{\theta} \mathbf{a}_3 \times L_1 (c\theta \mathbf{a}_1 + s\theta \mathbf{a}_2) = -L_1 \dot{\theta} s\theta \mathbf{a}_1 + L_1 \dot{\theta} c\theta \mathbf{a}_2
\]

The position vector of point \( P \) relative to point \( B \) is

\[
\mathbf{r}_{P/B} = L_2 c\phi \mathbf{a}_1 - L_2 s\phi \mathbf{a}_2
\]

Thus we have

\[
\mathbf{v}_P = \mathbf{v}_B + (-\dot{\phi} \mathbf{a}_3) \times L_2 (c\phi \mathbf{a}_1 - s\phi \mathbf{a}_2) = \mathbf{v}_B - L_2 \dot{\phi} s\phi \mathbf{a}_1 - L_2 \dot{\phi} c\phi \mathbf{a}_2
\]

Evaluating (17) and noting that the velocity of \( P \) must be along the \( \mathbf{a}_1 \) direction yields

\[
\mathbf{v}_P = -L_1 \dot{\theta} s\theta + L_2 \dot{\phi} s\phi \mathbf{a}_1 + (L_1 \dot{\theta} c\theta - L_2 \dot{\phi} c\phi) \mathbf{a}_2 = v_P \mathbf{a}_1
\]

Since the \( \mathbf{a}_2 \) component of (18) must be zero, we have

\[
L_1 \dot{\theta} c\theta = L_2 \dot{\phi} c\phi
\]

Solving for \( \dot{\phi} \) from (19) and substituting, we obtain the velocity of \( P \) as

\[
\mathbf{v}_P = -L_1 \dot{\theta} (\sin \theta + \cos \theta \tan \phi) \mathbf{a}_1
\]

The virtual displacement comes from the velocity, and is

\[
\delta \mathbf{r}_p = -L_1 (\sin \theta + \cos \theta \tan \phi) \delta \theta \mathbf{a}_1
\]

where (using trigonometry) it can be shown that

\[
\tan \phi = \frac{L_1 \sin \theta}{\sqrt{L_2^2 - L_1^2 \sin^2 \theta}}
\]

(b) Analytical Method. Here we find the position \( \mathbf{r}_P = \mathbf{r}_B + \mathbf{r}_{P/B} \):

\[
\mathbf{r}_P = (L_1 c\theta + L_2 c\phi) \mathbf{a}_1 + (L_1 s\theta - L_2 s\phi) \mathbf{a}_2
\]

Since the \( \mathbf{a}_2 \) component must be zero, we get

\[
L_1 s\theta = L_2 s\phi
\]

which is just the law of sines! The \( \mathbf{a}_1 \) component yields

\[
\mathbf{r}_P = (L_1 c\theta + L_2 c\phi) \mathbf{a}_1
\]

Taking the variation of (25) gives

\[
\delta \mathbf{r}_P = \frac{\partial \mathbf{r}_P}{\partial \theta} \delta \theta + \frac{\partial \mathbf{r}_P}{\partial \phi} \delta \phi = (-L_1 s\theta \mathbf{a}_1 - L_2 s\phi \mathbf{a}_2)
\]

But taking the variation of (24) yields

\[
L_1 c\theta \delta \theta = L_2 c\phi \delta \phi \quad \Rightarrow \quad \delta \phi = \frac{L_1 c\theta}{L_2 c\phi} \delta \phi
\]

Substituting (27) into (26) yields the same result as before (and perhaps with less work!),

\[
\delta \mathbf{r}_P = -L_1 (\sin \theta + \cos \theta \tan \phi) \delta \theta \mathbf{a}_1
\]
**Problem 4.11.** Here you already have the virtual displacement $\delta r_p$ from Problem 4.8, this is needed to find the virtual work done by nonconservative force $F$. The generalized forces associated with the gravitational forces may be found using virtual work or the potential function $V$ (that’s what I will use).

*Potential Function.* The c.g. of both links rises the same amount, so

$$V = \frac{1}{2} (m_1 + m_2) g L_1 \sin \theta$$

(29)

*Nonconservative force.* The virtual work done by $F$ is

$$F \mathbf{a}_1 \cdot \delta r_p = - F L_1 (\sin \theta + \cos \theta \tan \phi) \delta \theta$$

(30)

Thus the total generalized force is

$$Q_\theta = - \frac{\partial V}{\partial \theta} + Q_{\theta nc} = -\frac{1}{2} (m_1 + m_2) g L_1 \cos \theta - F L_1 (\sin \theta + \cos \theta \tan \phi)$$

(31)

where again

$$\tan \phi = \frac{L_1 \sin \theta}{\sqrt{L_2^2 - L_1^2 \sin^2 \theta}}$$

**Problem 4.13.** First do the conservative (gravitational) forces; the potential function is

$$V = 2 mg L^2 \sin \theta + mg[(L - R\phi) \sin \theta + R \cos \theta]$$

(32)

$$= 2 mg L \sin \theta + mg(R \cos \theta - \phi \sin \theta)$$

(33)

The conservative generalized forces are

$$Q_\theta c = - \frac{\partial V}{\partial \theta} = - 2 mg L \cos \theta + mg(R \sin \theta + \phi \cos \theta)$$

(34)

$$Q_\phi c = - \frac{\partial V}{\partial \phi} = mg R \sin \theta$$

(35)

There are two nonconservative forces, force $F$ and moment $M$. Force $F$ is applied at point $G$, so we can use the virtual displacement $\delta r_G$ found in Example 4.5, (equation [e] p. 234). Moment $M$ is applied around point $B$.

The nonconservative virtual work is given by

$$\delta W_{nc} = F(\cos \psi i + \sin \psi j) \cdot [ - R(\delta \phi + \delta \theta) i + (L - R\phi) \delta \theta j] + M k \cdot \delta k$$

(36)

$$= - FR \cos \psi (\delta \phi + \delta \theta) + F \sin \psi (L - R\phi) \delta \theta + M \delta \theta$$

(37)

Thus

$$Q_{\theta nc} = - FR \cos \psi + F \sin \psi (L - R\phi) + M$$

(38)

$$Q_{\phi nc} = - FR \cos \psi$$

(39)

The total generalized forces are:

$$Q_\theta = - FR \cos \psi + F \sin \psi (L - R\phi) + M - 2 mg L \cos \theta + mg R (\sin \theta + \phi \cos \theta)$$

(40)

$$Q_\phi = - FR \cos \psi + mg R \sin \theta$$

(41)

**Problem 4.14.** It is useful to define angle $\phi$ as shown in Figure 4 on the next page. All forces in this problem are conservative, and $\theta$ is the single generalized coordinate, so we can find static equilibrium by setting $\partial V/\partial \theta = 0$, where $V$ is the system potential function.

Let $x$ be the elongation of the spring—then the potential function $V$ is (note that the c.g. of both links moves down the same amount):

$$V = - \left( m + \frac{m}{2} \right) g \frac{L}{2} \sin \theta + \frac{1}{2} k x^2$$

(42)

$$= - \frac{3}{4} mg L \sin \theta + \frac{1}{2} k x^2$$

(43)
Spring elongation \(x\) is given by
\[
x = \frac{3}{2} L - L \cos \theta - \frac{L}{2} \cos \phi \tag{44}
\]
\[
= L(1 - \cos \theta) + \frac{L}{2} (1 - \cos \phi) \tag{45}
\]

At equilibrium, using the chain rule on the spring potential function, we have
\[
\frac{\partial V}{\partial \theta} = -\frac{3}{4} mgL \cos \theta + kx \frac{\partial x}{\partial \theta} = 0 \tag{46}
\]

where
\[
\frac{\partial x}{\partial \theta} = L \sin \theta + \frac{L}{2} \sin \phi \frac{\partial \phi}{\partial \theta} \tag{47}
\]

However, \(\phi\) and \(\theta\) can be related using the Law of Sines, and using the differential we can find \(\partial \phi / \partial \theta\) needed in (47):
\[
2 \sin \theta = \sin \phi \implies 2 \cos \theta \, d\theta = \cos \phi \, d\phi \implies \frac{\partial \phi}{\partial \theta} = \frac{d\phi}{d\theta} = 2 \cos \theta \cos \phi \tag{48}
\]

Substituting (48) and (47) into (46), canceling the \(L\), and after a little trigonometric reduction we get the equilibrium condition as
\[
\frac{3}{4} mgL \cos \theta - kxL \left[\sin \theta + \cos \theta \tan \phi\right] = 0 \tag{49}
\]

where \(\phi\) can be related to \(\theta\) using (48) and \(x\) can be related to \(\theta\) and \(\phi\) using (45).

**Extra Credit.** With \(m = 1\) kg, \(L = 1\) m, \(g = 9.81\) m/s\(^2\), the MATLAB `fzero` function can be used to find the numerical equilibrium solution. The M-file below is stored as `prob14.m`.

```matlab
function f = prob14(theta)

% This function computes the dV/dtheta function for Problem 4.14. We have
% to calculate 'phi' and 'x' since they are used in the function.

m = 1; % mass in kg
L = 1; % length in m
g = 9.81; % gravitational constant in m/s^2
k = 100; % spring constant in N/m

phi = asin(2*sin(theta)); % Compute angle 'phi'
x = L*(1-cos(theta))+(L/2)*(1-cos(phi)); % Compute spring elongation
f = -(3/4)*m*g*cos(theta)+k*x*(sin(theta)+cos(theta)*tan(phi)); % Compute function
end
```

Executing the `fzero` function, we get
\[
>> \texttt{fzero('fun',0.1)}
\]
\[
\text{ans} = 0.2440 \quad (13.9809 \text{ degrees})
\]