3.9 A certain two-dimensional incompressible, steady flow is given by the stream function
\[ \phi = A \ln r + B\theta, \]
where \( r \) and \( \theta \) are the usual polar coordinates and \( A \) and \( B \) are positive constants.

(a) Find the components of fluid velocity and show that the continuity equation is satisfied.

(b) Sketch a sufficient number of streamlines so that the flow pattern becomes clearly evident.

(c) Find the radial and tangential components of fluid acceleration.

(d) Find the distribution of pressure as a function of \( r \) and \( \theta \) by two methods (i.e., starting with the differential equation of motion directly and using the Bernoulli equation).

3.15 A circular plate is forced down at a steady velocity \( V_0 \) against a flat surface. Frictionless fluid of density \( \rho \) fills the gap \( h \). Assume that \( h \ll R_0 \), the plate radius, and that the radial velocity \( V_r(R, t) \) is constant across the gap.

PROBLEM 3.15

(a) From continuity considerations, obtain a formula for \( V_r(R, t) \) in terms of \( R, V_0 \), and \( h \).

(b) Noting that \( h = h(t) \), evaluate \( \partial V_r / \partial t \).

(c) Substitute into the Bernoulli equation and calculate the pressure distribution, assuming that \( p(R = R_0, t) = 0 \).
\[ \psi = A \ln r + B \Theta \]

\[ \psi = \frac{\partial \psi}{\partial \Theta} = -\frac{B}{r} \]

\[ \frac{B}{r} = -\frac{\partial \psi}{\partial r} = -\frac{A}{r} \]

\[ \nabla \cdot \mathbf{v} = -\frac{1}{r} \frac{2}{2r} (r^2 \nabla \psi) + \frac{1}{r} \frac{2}{\Theta} \frac{\partial \psi}{\partial \Theta} \]

\[ = -\frac{1}{r} \frac{2}{\Theta} (B) + \frac{1}{r} \frac{2}{\Theta} (-\frac{A}{r}) \]

\[ = 0 \quad \text{continuity satisfied} \]

b) lines of constant \( \psi \) (streamlines) require that:

\[ A \ln r = \psi - B \Theta \]

\[ r = e^{\frac{B}{A}} e^{-\frac{B}{A} \Theta} \]
a) The acceleration of a fluid particle is a Lagrangian quantity:

\[ a = \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \nabla \mathbf{u} \]

The flow is steady, so \( \frac{\partial \mathbf{u}}{\partial t} = 0 \)

\[ \nabla \cdot \mathbf{u} = \frac{\partial u_r}{\partial r} = -\frac{B}{r^2} \]

\[ \nabla \cdot \mathbf{u} = \frac{\partial u_\theta}{\partial \theta} = \frac{A}{r^2} \]

\[ \nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} = \frac{A}{r^2} \]

\[ \nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{B}{r^2} \]

\[ \mathbf{u} \cdot \nabla \mathbf{v} = \left( u_r \nabla u_r + u_\theta \nabla u_\theta \right) \hat{r} + \left( u_r \nabla u_\theta + u_\theta \nabla u_r \right) \hat{\theta} \]

\[ = \left( -\frac{A^2}{r^2} - \frac{B^2}{r^2} \right) \hat{r} + \left( \frac{AB}{r^2} - \frac{AB}{r^3} \right) \hat{\theta} \]

The tangential component can be found by dotting with the tangent vector. This, in turn, can be found by dividing \( \mathbf{v} \) by its magnitude, \( |\mathbf{v}| \).
\[ |\nabla| = \frac{\sqrt{A^2 + B^2}}{r} \]

\[ \nabla = \frac{\nabla}{|\nabla|} = \frac{r}{\sqrt{A^2 + B^2}} \left( \frac{B}{r} \delta r - \frac{A}{r} \delta \theta \right) \]

\[ = \frac{1}{\sqrt{A^2 + B^2}} \left( B \delta r - A \delta \theta \right) \]

\[ a_\theta = (\nabla \cdot \nabla) \cdot \nabla \]

\[ = -\left( \frac{A^2 + B^2}{r^2} \right) \cdot \frac{B}{\sqrt{A^2 + B^2}} = -\frac{B \sqrt{B^2 + A^2}}{r^3} \]

\[ a_r = \sqrt{|a_\theta|^2} - a_c^2 \]

\[ = \left[ \left( \frac{B^2 + A^2}{r^3} \right)^2 - \frac{B^2 (B^2 + A^2)}{r^6} \right]^{\frac{1}{2}} \]

\[ = \frac{A \sqrt{B^2 + A^2}}{r^3} \quad \text{(towards center of curvature)} \]
a) \[ \omega_2 = \frac{\partial^2 \rho}{\partial r^2} + \frac{1}{r} \frac{\partial \rho}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \rho}{\partial \theta^2} \]

\[ = -\frac{A}{r^2} + \frac{A}{r^2} + 0 \]

\[ = 0 \]

The flow is irrotational - we can use the Bernoulli equation across streamlines!

Say that pressure is known at \( \infty \).
At \( \infty \), \( u_r = 0 \) and \( u_\theta = 0 \)

\[ \frac{p_\infty}{\rho} = \left[ \frac{u(r, \theta)}{2} \right]^2 + \frac{p}{\rho} \]

\[ u(r, \theta) = \frac{\sqrt{a^2 + b^2}}{r} \]

\[ p = p_\infty - \frac{(a^2 + b^2)\rho}{2r^2} \]

Now use momentum equations:
Note: the flow is irrotational, therefore inviscid, so we can neglect all the viscous stresses. Also, we can neglect gravity forces.
\( \theta \) - direction:
\[
\frac{\partial p}{\partial \theta} = -\rho \left( \frac{\partial}{\partial \theta} \left( \frac{\partial \theta}{\partial \theta} \right) + \frac{\partial \theta}{\partial \theta} \right)
= -\rho \left( \frac{A^2}{R^3} - \frac{A^2}{R^3} \right) = 0
\]

- no \( \theta \) dependence, as expected from previous results using Bernoulli.

\( r \) - direction:
\[
\frac{\partial p}{\partial r} = -\rho \left( \frac{\partial}{\partial r} \left( \frac{\partial \theta}{\partial r} \right) - \frac{\partial \theta}{\partial \theta} \right)
= -\rho \left( \frac{B^2}{R^3} - \frac{B^2}{R^3} - \frac{A^2}{R^3} \right)
= \rho \frac{A^2}{R^3} (B^2 + A^2)
\]

\( p_{\infty} - p_{\theta \theta} = \int_{r}^{\infty} \frac{\rho}{R^3} (B^2 + A^2) dr \)
\[
= -\frac{1}{2} \rho \left( B^2 + A^2 \right) \left( \frac{1}{R^2} \right) \bigg|_{r}^{\infty}
= \frac{1}{2} \rho \frac{(B^2 + A^2)}{R^2}
\]
\[ p_{r,0} = p_0 - \frac{1}{2} \rho \frac{(b^2 + h^2)}{r^2} \]

Same as with Bernoulli.

3.15

Volume enclosed by \( r = \pi r^2 h \)

Rate of change of volume \( \frac{\partial V}{\partial t} = -\pi r^2 \frac{dh}{dt} \)

Fluid loss through circumference \( = 2\pi rhV_r \)

So \( 2\pi rhV_r = \pi r^2 V_0 \)

\[ V_r = \frac{V_0 r}{2h} \]

\( \frac{\partial V_r}{\partial t} = \frac{\partial V_r}{\partial h} \cdot \frac{\partial h}{\partial t} = -\frac{V_0 r}{2h^2} \cdot V_r = \frac{V_0^2 r}{2h^2} \)
c) B. equation with unsteady term:

\[ B(t) = \frac{\partial \rho}{\partial t} \cdot \sum r + \frac{V^2}{2} + \frac{p_r}{p} = \text{constant} \]

\[ \frac{V^2}{2} + \frac{p_r}{p} = \frac{V_{\infty}^2}{2} + \frac{P_{\infty}}{p} + \int_{r}^{R_0} \frac{\partial \rho}{\partial t} \cdot dr \]

\[ V_{\infty} = \frac{V_0 R_0}{2h} \]

\[ P_{\infty} = 0 \]

\[ \int_{r}^{R_0} \frac{\partial V}{\partial t} \cdot dr = \int_{r}^{R_0} \frac{V_0^2 - V^2}{2h^2} \cdot dr = \left[ \frac{V_0^2 r^2}{4h^2} \right]_{r}^{R_0} \]

\[ = \frac{V_0^2 R_0^2}{4h^2} - \frac{V_0^2 r^2}{4h^2} \]

\[ \frac{V_0^2 r^2}{8h^2} + \frac{p_r}{p} = \frac{V_0^2 R_0^2}{8h^2} + \frac{V_0^2 R_0^2}{4h^2} - \frac{V_0^2 r^2}{4h^2} \]

\[ p_r = \frac{3}{8} \frac{V_0^2}{h^2} (R_0^2 - r^2) \rho \]