\[ a) \int_{FR} \rho n_x \, dV = \int_{\frac{\pi}{2}}^{\pi} \int_{r_0}^{R_0} \rho \left( -n_r \cos \theta - n_\theta \sin \theta \right) \, dr \, d\theta \]

\[ = -\rho U \int_{\frac{\pi}{2}}^{\pi} \int_{r_0}^{R_0} \cos \theta \, r \left[ 1 - \left( \frac{r_0}{r} \right)^2 \right] \, dr \, d\theta \]

\[ + \rho U \int_{\frac{\pi}{2}}^{\pi} \int_{r_0}^{R_0} \sin^2 \theta \, r \left[ 1 + \left( \frac{r_0}{r} \right)^2 \right] \, dr \, d\theta \]

\[ = -\rho U \int_{\frac{\pi}{2}}^{\pi} \int_{r_0}^{R_0} r \, dr \, d\theta + \rho U \int_{\frac{\pi}{2}}^{\pi} \int_{r_0}^{R_0} \cos^2 \theta \, r \frac{r_0^2}{r} \, dr \, d\theta \]

\[ = -\rho U \int_{\frac{\pi}{2}}^{\pi} \int_{r_0}^{R_0} \sin^2 \theta \, \frac{r_0^2}{r} \, dr \, d\theta \]
\[
\begin{align*}
&= - \rho \frac{U \pi}{2} \left( R_1^2 - R_0^2 \right) + \int_0^{\frac{\pi}{2}} \rho U \ln \left( \frac{R_0}{r_1} \right) \cos \theta \ d\theta \\
&= - \rho \frac{U \pi}{2} \left( R_1^2 - R_0^2 \right) = \text{x-momentum of volume defined by fr.}
\end{align*}
\]

b) \[
\int_{S_1} \mathbf{p} \cdot \mathbf{n} \ ds = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \rho U \mathbf{n} \left. \right|_{r=r_0} \ d\theta
\]

\[
= - \rho U r_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta \left[ 1 - \left( \frac{R_0}{r_0} \right)^2 \right] \ d\theta
\]

\[
= - \rho U r_0 \left[ 1 - \left( \frac{R_0}{r_0} \right)^2 \right] \sin \theta \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
\]

\[
= -2 \rho U r_0 \left[ 1 - \left( \frac{R_0}{r_0} \right)^2 \right]
\]

\[
= \text{mass flow rate through } S_1
\]
c) \[ \int \nabla \cdot (\rho \mathbf{v}) \mathbf{s} \, ds = \int_{S_2} \frac{1}{2} \rho U^2 \left[ \frac{2}{1} \left( \frac{r}{R} \right)^2 \left( 1 - 2 \sin^2 \frac{\pi}{6} \right) - \left( \frac{r}{R} \right)^4 \right] dr \]
\[ = \frac{1}{2} \rho U^2 \left[ \frac{2}{3} \left( \frac{R}{r} \right)^3 \right] \]
\[ = \frac{1}{2} \rho U^2 \left[ \frac{2}{3} \frac{R^2}{r^2} + \frac{\frac{r^4}{3}}{R} - \left( \frac{r}{R} \right)^3 \right] \]
\[ = \frac{1}{2} \rho U^2 r \left[ -2 \frac{r}{R} + \frac{1}{3} \left( \frac{r}{R} \right)^3 - \frac{7}{3} \right] \]
\[ = x \cdot \text{force on } S_2 \]

d) \[ \int_{S_2} \rho \nabla \cdot \mathbf{v} \mathbf{s} \, ds = \int_{S_2} -\rho r \mathbf{v} \cdot \mathbf{n} \, dr \]
\[ = \rho \int_{R_0}^{R_0} U^2 \sin^2 \left( \frac{\pi}{2} \right) \left[ 1 + \left( \frac{r}{R} \right)^2 \right]^2 dr \]
\[ = \rho U^2 \int_{R_0}^{R_0} \left[ 1 + \left( \frac{r}{R} \right)^2 \right] \left[ 1 + \left( \frac{r}{R} \right)^2 \right] dr \]
\[ \text{set } \frac{R}{r} = \xi \]
\[ dr = -R_0 \xi^{-2} d\xi \]
\[ \begin{align*}
  &= \rho u^2 \int_{r_0}^{R_0} \left[ 1 + \xi^2 \right] \xi^{-2} \, d\xi \\
  &= -\rho u^2 \int_{r_0}^{R_0} \left( \xi^{-2} + \xi^2 + 2 \right) \, d\xi \\
  &= -\rho u^2 \xi \left[ -\xi^{-1} + \frac{1}{3} \xi^3 + 2 \xi \right]_{r_0}^{R_0} \\
  &= \rho u^2 \left[ -R_0^{-1} + \frac{1}{3} R_0^3 + 2 R_0 - 4 r_0 \right] \\
  &= x - \text{direction momentum crossing } S_2
\end{align*} \]

e) \quad -\int_{S_3} n_x p \, dS = -\frac{R_0}{2} \rho u^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \theta - 4 \sin^2 \theta \cos \theta) \, d\theta \\
\quad = -\frac{R_0}{2} \rho u^2 \left[ \sin \theta - \frac{4}{3} \sin^3 \theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
\quad = \frac{2}{3} \frac{R_0}{2} \rho u^2 \quad x - \text{direction force on front surface of cylinder}
\( \int_{S_3} \rho u d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} \rho u^2 (1 - 4 \sin^2 \theta) r \, d\theta \)

\[ = -\frac{1}{2} \rho u^2 r_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - 4 \sin^2 \theta) \, d\theta \]

\[ = -\frac{1}{2} \rho u^2 r_0 \left[ \theta + 2 \cos \theta \sin \theta - 2 \theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \]

\[ = \frac{1}{2} \rho u^2 r_0 \pi \]

= average pressure of curved surface \( S_3 \).
5.2

For constant $p$, continuity equation is

$$\nabla \cdot \mathbf{N} = 0$$

In spherical coordinates,

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 n_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (n_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} n_\phi = 0$$

$n_r = n_r(r)$

$n_\theta = 0$

$n_\phi = 0$

So \( \frac{\partial}{\partial r} (r^2 n_r) = 0 \) or \( r^2 n_r = C \)

In cylindrical coordinates:

$$\frac{1}{r} \frac{\partial}{\partial r} (r n_r) + \frac{1}{\partial \theta} (n_\theta) + \frac{\partial}{\partial z} n_z = 0$$

$n_r = n_r(r)$, $n_\theta = 0$, $n_z = 0$

$$\frac{\partial}{\partial r} (r n_r) = 0$$ or \( r n_r = D \)
5.7

Show that the kinetic energy equation can be derived from the momentum equation and the continuity equation:

**KE eqn:**

\[
\frac{\partial}{\partial t} \left( \rho \frac{1}{2} v^2 \right) + \nabla \cdot \left( \rho \frac{1}{2} v^2 \right) = -\nabla \cdot \rho \mathbf{v} + \nabla \cdot \left( \rho \mathbf{a} \right) + \rho \mathbf{F}
\]

**Mom. Eqa:**

\[
\frac{2}{\partial t} (p v) + \nabla \cdot (p v \mathbf{v}) = -\nabla p + \boldsymbol{v} \cdot \xi + \rho \mathbf{F}
\]

**Cont. Eqa:**

\[
\frac{\partial e}{\partial t} + \nabla \cdot (p v) = 0
\]
Clearly, the RHS of the KE equation is simply the momentum eqn. dotted with \( \mathbf{v} \). We need to show that the LHS of the KE eqn. is the same as the LHS of the momentum equation dotted with \( \mathbf{v} \).

Take 1st term on LHS of KE:

\[
\frac{\partial}{\partial t} \left( \rho \frac{1}{2} \mathbf{v}^2 \right) = \frac{\partial}{\partial t} \left( \rho \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) 
\]

\[
= \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \rho}{\partial t} \frac{1}{2} \mathbf{v} \cdot \mathbf{v} 
\]

\[
= \rho \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{v} + \frac{1}{2} \frac{\partial \rho}{\partial t} \mathbf{v} \cdot \mathbf{v} 
\]

Now look at 2nd term:

\[
\nabla \cdot \left( \rho \frac{1}{2} \mathbf{v}^2 \right) = \nabla \cdot \left( \rho \mathbf{v} \cdot \frac{1}{2} \mathbf{v} - \mathbf{v} \right) 
\]

\[
= \frac{1}{2} \nabla \cdot \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \rho \mathbf{v} \cdot \nabla \mathbf{v} 
\]

Now combine both terms:

\[
\frac{\partial}{\partial t} \left( \rho \frac{1}{2} \mathbf{v}^2 \right) + \nabla \cdot \left( \rho \frac{1}{2} \mathbf{v}^2 \right) = 
\]

\[
= \rho \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{v} + \frac{1}{2} \frac{\partial \rho}{\partial t} \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \rho \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{2} \rho \mathbf{v} \cdot \nabla \mathbf{v} 
\]
From continuity, we know that \( \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \), so that we can add the term

\[
\frac{1}{2} \frac{\partial \rho}{\partial t} \mathbf{v} \cdot \mathbf{v} \quad + \quad \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \cdot \frac{\partial \rho}{\partial t} + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \cdot \nabla \cdot \mathbf{v}
\]

to our expression without disturbing it:

\[
\rho \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{v} + \frac{\partial \rho}{\partial t} \mathbf{v} \cdot \mathbf{v} + \frac{\partial \rho}{\partial t} \rho \mathbf{v} + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \cdot \nabla \cdot \mathbf{v}
\]

\[
= \frac{\partial \rho}{\partial t} \mathbf{v} \cdot \mathbf{v} + \frac{\partial \rho}{\partial t} \rho \mathbf{v} + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \cdot \nabla \cdot \mathbf{v}
\]

\[
= \nabla \cdot \left[ \frac{\partial \rho}{\partial t} \mathbf{v} + \rho \mathbf{v} \cdot \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} \right]
\]

\[
= \nabla \cdot \left[ \frac{\partial \rho}{\partial t} \mathbf{v} + \mathbf{v} \cdot \nabla (\rho \mathbf{v}) \right]
\]

This is the LHS of the momentum equation, dotted with the \( \mathbf{v} \) vector. @ED.