t-test comparison of sample means (3.6.4)

Suppose we have two samples: 1, 2 with

\[ \bar{x}_1, s_1, n_1 \quad \bar{x}_2, s_2, n_2. \]

Approx. degree of freedom (assuming \( n_1 \neq n_2 \)) has to be calculated thus (3.25a)

\[ v = \left( \frac{s_1^2/n_1 + s_2^2/n_2}{\frac{(s_1^2/n_1)^2}{n_1} + \frac{(s_2^2/n_2)^2}{n_2}} \right)^\frac{1}{2} \]

Round off to nearest integer

Then the formula for sample comparison is (3.25)

Test statistic \( t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_1^2/n_1 + s_2^2/n_2}} \)

What does \( t \) mean?

\[ t \text{ falls inside } \pm t_{d/2, p} \Rightarrow (3.25a) \]

samples are not significantly different @ chosen level of confidence L
Dealing with Uncertainty. (3.3.3.39)

Some assumptions:

- for statistically independent sources of error $\epsilon_1$ and $\epsilon_2$, the cumulative error for a large # of samples is
  \[ \epsilon = \sqrt{\epsilon_1^2 + \epsilon_2^2} \]

- bias and precision error are statistically independent

- uncertainty due to instrument error and due to spread in measurements are statistically independent

Under these assumptions...

Precision uncertainty in $x$

at a confidence level $c\%$

\[ p_x = t \frac{1}{n^2} \frac{S_x}{\sqrt{n}} \] (t-statistics, eqn. 3.24)

Bias uncertainty has to be found by calibration: $B_x$

Total uncertainty $\ U_x = \sqrt{B_x^2 + p_x^2}$

$B_x$ and $p_x$ must be estimates for same coverage.

$B_x, p_x$ - both for 95% $\Rightarrow \ U_x$ $\Rightarrow$ for 95%

\[ N^3 \ (strange \ but \ true) \]

$B_x, p_x$ - 95% coverage : $B_x + p_x = $ uncertainty with 99% coverage.
Propagation of uncertainty (3.10)

1. $y = y(x_1, \ldots, x_n)$, $y$ can be linearized near $(x_1 = 0, \ldots, x_n = 0)$.

2. For each $x_i$, $\sigma_i$ is the standard deviation.

$1+2 \Rightarrow$ Standard deviation of $y$

$$\sigma_y = \sqrt{\left(\frac{\partial y}{\partial x_1} | \sigma_1 \right)^2 + \left(\frac{\partial y}{\partial x_2} | \sigma_2 \right)^2 + \cdots + \left(\frac{\partial y}{\partial x_n} | \sigma_n \right)^2}.$$ 

This result - exact if $y$ is a linear function.

Uncertainties behave in a way similar to standard deviations.

Let $x_1, \ldots, x_n$ be something we measure.

$u_1, \ldots, u_n$ - corresponding measurement uncertainty.

$y(x_1, \ldots, x_n)$ - function of measured variables.

$u_y$ - its uncertainty.

Let $y$ have a Taylor series expansion:

$$y(x_1 + u_1, x_2 + u_2, \ldots, x_n + u_n) =$$

$$= y(x_1, \ldots, x_n) + u_1 \frac{\partial y}{\partial x_1} |_{x_1, \ldots, x_n} + u_2 \frac{\partial y}{\partial x_2} |_{x_1, \ldots, x_n} + \cdots + u_n \frac{\partial y}{\partial x_n} |_{x_1, \ldots, x_n} + \text{higher-order terms.}$$

High-order terms negligible $\Rightarrow$ $y$ - linearizable in $u_1, \ldots, u_n.
Then
\[ u_y = \sqrt{\left(\frac{\partial y}{\partial x_1} u_{1}\right)^2 + \ldots + \left(\frac{\partial y}{\partial x_n} u_n\right)^2}. \]

Example. \( x \) is measured with \( \pm 5\% \) uncertainty. \( (1 \leq x \leq 3). \)

What is the uncertainty in
\[ y = 0.3x^2 - 0.1x? \]

Solution. \[ \frac{\partial y}{\partial x} = 0.3 \cdot 2x - 0.1 = 0.6x - 0.1 \]

\[ \frac{d x}{x} = 0.05. \]

\[ u_y = \sqrt{\left(\frac{\partial y}{\partial x} u_x\right)^2} = \left| \frac{\partial y}{\partial x} u_x \right|. \]

\[ \left| \frac{\partial y}{\partial x} u_x \right| = \left| (0.6x - 0.1) \cdot 0.05x \right| =
\]

\[ = \left| 0.03x^2 - 0.005x \right|
\]

\[ \left| \frac{\partial y}{\partial x} u_x \right|_{\text{max}} = (0.03 \cdot 3^2 - 0.005 \cdot 3) = 0.255. \]

\[ u_y \text{ max} = \pm 0.255 \]
\( \chi^2 \) statistics.

**Remark.** For a \( c\% \) confidence interval, the boundaries are

\[
\left( \bar{x} - Z_{c/2} \frac{S}{\sqrt{n}}, \bar{x} + Z_{c/2} \frac{S}{\sqrt{n}} \right) \quad \text{or...}
\]

\[
\left( \bar{x} - Z_{c/2} \frac{S_x}{\sqrt{m}}, \bar{x} + Z_{c/2} \frac{S_x}{\sqrt{m}} \right)
\]

(3.19)

\( (n \text{ sample, } S_x = \text{st. dev, } \bar{x} = \text{mean}) \).

\[
\frac{S_x}{\sqrt{n}} = \text{called standard error.}
\]

(3.19): good when we have a sample mean and want uncertainty for population mean.

OTOH, with a normal distribution (Table 3.1) the confidence interval for a measurement is...

90% \( \pm 1.645 \)

95% \( \pm 1.960 \)

99.9% \( \pm 3.29 \) \( \text{both called "maximum error."} \)

Don't ask why.

(3.19): confidence interval for mean. Can we form a confidence interval for standard deviation?

Yes, with \( \chi^2 \)-distribution (f. 3.5)
\[ \frac{(n-1) S_x^2}{\chi^2_{n-1}} < \sigma^2 < \frac{(n-1) S_x^2}{\chi^2_{0.5}} \quad (c\% = 1-\alpha) \]

NB: \( \chi^2 \) - not symmetric.

- can be used to test sample normally. (NB: 3.30)

What is \( \chi^2 \)?

Example: A random-number generator churns out 100 numbers 1 to 9 with the distribution as listed. Assess if the distribution differs significantly from expected @ 95% confidence level.

<table>
<thead>
<tr>
<th>Digit</th>
<th>Observed freq.</th>
<th>Expected freq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>18</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

Solution

Random-number generator should spit out every digit with equal probability thus the "expected" distribution should be 100/10 = 10 per each.
\[ \chi^2 = \frac{(7-10)^2}{10} + \frac{(12-10)^2}{10} + \frac{(12-10)^2}{10} + \frac{(7-10)^2}{10} + \frac{(6-10)^2}{10} + \frac{(8-10)^2}{10} + \frac{(4-10)^2}{10} = 8.6. \]

\[ \chi = 1 - 99\% = 1\% = 0.01. \]

\[ \nu = 10 - 1 = 9. \]

Look up \( \chi^2_{0.01}, 9 \) in \( \chi^2 \) table: \( \chi^2_{0.01} = 21.66 \)

\[ 8.6 < 21.66 \Rightarrow \text{no significant difference} \]

Ans.