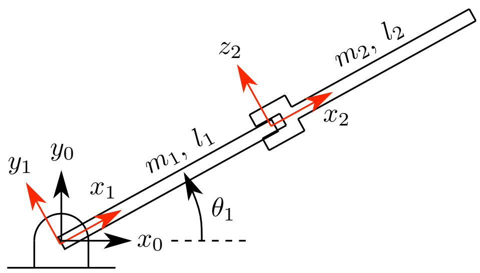
Chapter 6 HW Solution

Starr Problem. The figure below shows the RR manipulator with attached link frames. There are also frames $\{C_1\}$ and $\{C_2\}$ at the center of the links (with the same orientation as frames $\{1\}$ and $\{2\}$).



The table of link parameters for this robot is

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-90°$	l_1	0	θ_2

and the two transformation matrices are

$$ {}^0_1\mathbf{T} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad {}^1_2\mathbf{T} = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

The inertia matrices for both links have the same structure (since in each case the y and z axes are normal to the link):

$$ I_C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{2}$$

where in each case $I = ml^2/12$.

I wrote a MATLAB script (which used the Symbolic Math Toolbox) to perform the outward kinematic iterations and inward force/moment iterations (6.45–6.53), and finally obtained the following matrices:

$$ \mathbf{M} = \begin{bmatrix} \frac{1}{3}m_1l_1^2 + \frac{1}{3}m_2l_2^2c_2^2 + m_2(l_1^2 + l_1l_2c_2) & 0 \\ 0 & \frac{1}{3}m_2l_2^2 \end{bmatrix}, \tag{3}$$

$$ \mathbf{V} = \begin{bmatrix} (-m_2l_1l_2s_2 - \frac{2}{3}m_2l_2^2s_2c_2)\dot{\theta}_1\dot{\theta}_2 \\ (\frac{1}{3}m_2l_2^2s_2c_2 + \frac{1}{2}m_2l_1l_2s_2)\dot{\theta}_1^2 \end{bmatrix}, \tag{4}$$

$$ \mathbf{G} = \begin{bmatrix} \frac{1}{2}\left(m_1 + m_2\right)gl_1c_1 + \frac{1}{2}m_2gl_2c_1c_2 \\ -\frac{1}{2}m_2gl_2s_1s_2 \end{bmatrix}. \tag{5}$$

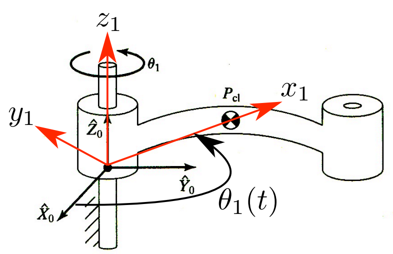
Note that mass matrix $\mathbf{M}$ is diagonal, indicating that there is *no dynamic coupling.* This means the inertia torque from one link does not affect the other links. There is coupling in other ways, however. Compare this $\mathbf{M}$ with the one from the Programming Exercises, where there *is* dynamic coupling.

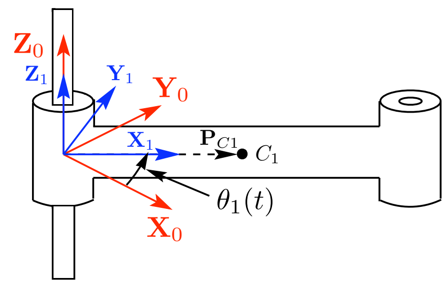
6.11. Since the $\hat{\mathbf{Z}}_1$ axis is the axis of rotation, the moment of inertia I_{zz1} is the inertia that is being rotated. From your control systems class, you learned that a gear train reduces the inertia "felt" by the motor by a factor of n^2, where n is the gear ratio ($n > 1$ for a speed reducing gear train). Hence the total inertia which the motor must accelerate is:

$$I_{total} = I_m + \frac{I_{zz1}}{n^2} = I_m + \frac{I_{zz1}}{100^2} = I_m + \frac{I_{zz1}}{10,000} \tag{6}$$

Thus the motor's own inertia may be dominant! (Gear ratios of ≈100 are not uncommon)

Problem 6.12. Refer to the figures below; frame {1} and joint angle $\theta_1(t)$ are shown. Since ${}^1P_C = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}^T$, we know that x_2 must pass through the center of mass.





The rotation matrix ${}^0_1\mathbf{R}$ is needed to find the reaction force on the support; this is a standard $\mathbf{R_Z}(\theta)$, given by

$${}^0_1\mathbf{R} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

The inertia matrix about the center of mass for the link can be expressed symbolically in frame {C} as

$${}^CI_1 = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \tag{8}$$

Note that since ${}^CI_1$ is diagonal then the axes of frame {1} must constitute *principal axes* for link 1. Since this implies that the xy, xz, and yz planes form *planes of symmetry* for the body, the poor drawing of the original figure is even more apparent! Oh, well...this is *still* a pretty good textbook.

(a) Find angular acceleration of link ${}^1\dot{\boldsymbol{\omega}}_1$ and accelertion of center of mass ${}^1\dot{\mathbf{v}}_{C_1}$.

Outward propagations: $i = 0 \rightarrow 1$. From inspection we can see that

$${}^0\omega_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \; {}^0\dot{\omega}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \; {}^0v_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \; {}^0\dot{v}_0 = \begin{bmatrix} 0 & 0 & g \end{bmatrix}^T \tag{9}$$

Note that the gravitation acceleration "g" is for purposes of modeling gravity and not really a kinematic acceleration. Using the propagation equations (6.45)-(6.50), I found that

$${}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \; {}^1\dot{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}, \; {}^1\dot{v}_1 = {}^1_0R\,{}^0\dot{v}_0 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \tag{10}$$

$${}^1\dot{v}_C = {}^1\dot{\omega}_1 \times {}^1P_C + {}^1\omega_1 \times \left({}^1\omega_1 \times {}^1P_C\right) + {}^1\dot{v}_1 = \begin{bmatrix} -l\dot{\theta}_1^2 \\ l\ddot{\theta}_1 \\ g \end{bmatrix} \tag{11}$$

(b) Find the required actuator torque $\boldsymbol{\tau}_1(t)$.

The resultant force and moment as

$$ {}^1F_1 = m\,{}^1\dot{v}_C = \begin{bmatrix} -ml\dot{\theta}_1^2 \\ ml\ddot{\theta}_1 \\ mg \end{bmatrix}, \quad {}^1N_1 = {}^CI_1\,{}^1\dot{\omega}_1 + {}^1\omega_1 \times \left({}^1\omega_1 \times {}^1P_C\right) = \begin{bmatrix} 0 \\ 0 \\ I_{zz}\ddot{\theta}_1 \end{bmatrix} \tag{12} $$

Inward propagations: The results I obtained are

$$ {}^1f_1 = {}^1F_1, \quad {}^1n_1 = {}^1N_1 + {}^1P_C \times {}^1F_1 = \begin{bmatrix} 0 \\ -mgl \\ \left(I_{zz} + ml^2\right)\ddot{\theta}_1 \end{bmatrix} \tag{13} $$

At this point we can find joint torque τ_1 as the Z component of ${}^1n_1$; this is

$$ \boxed{\tau_1 = \left(I_{zz} + ml^2\right)\ddot{\theta}_1} \tag{14} $$

(c) Reaction force and moment on the support. The reaction force and moment **on** the support are:

$-{}^1\mathbf{f}_1 =$ force exerted on link 0 (support) by link 1

$-{}^1\mathbf{n}_1 =$ moment exerted on link 0 (support) by link 1

However, we would like to express these in frame $\{0\}$. This can be done with the $\mathbf{R}$ matrix,

$$ \boxed{-{}^0\mathbf{f}_1 = -{}^0_1\mathbf{R}\,{}^1\mathbf{f}_1 = -\begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -ml\dot{\theta}_1^2 \\ ml\ddot{\theta}_1 \\ mg \end{bmatrix} = \begin{bmatrix} ml\dot{\theta}_1^2 c_1 + ml\ddot{\theta}_1 s_1 \\ ml\dot{\theta}_1^2 s_1 - ml\ddot{\theta}_1 c_1 \\ -mg \end{bmatrix}} \tag{15} $$

$$ \boxed{-{}^0\mathbf{n}_1 = -{}^0_1\mathbf{R}\,{}^1\mathbf{n}_1 = -\begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -mgl \\ \left(I_{zz} + ml^2\right)\ddot{\theta}_1 \end{bmatrix} = \begin{bmatrix} -mgls_1 \\ -mglc_1 \\ -\left(I_{zz} + ml^2\right)\ddot{\theta}_1 \end{bmatrix}} \tag{16} $$

Problem 6.14. The incorrect terms in the EOMs for the RP manipulator I found were:

$$ \begin{aligned} \tau_1 &= m_1(d_1^2 + d_2)\ddot{\theta}_1 + m_2 d_2^2\ddot{\theta}_1 + 2m_2 d_2\dot{d}_2\dot{\theta}_1 \\ &\quad + g\cos(\theta_1)[m_1(d_1 + d_2\dot{\theta}_1) + m_2(d_2 + \dot{d}_2)] \\ f_2 = \tau_2 &= m_1\dot{d}_2\ddot{\theta}_1 + m_2\ddot{d}_2 - m_1 d_1\dot{d}_2 - m_2 d_2\dot{\theta}^2 + m_2(d_2 + 1)g\sin(\theta_1) \end{aligned} $$

My reasoning was based on each term having the correct units of (M L^2)/T^2 (*i.e.* torque) in the first EOM, and units of (M L)/T^2 (*i.e.* force) in the second EOM.