DC Gain of Systems with an Integration

1 The Problem

There are situations—like my unfortunate choice of example in class on Wednesday for a demonstration of the “Pole-Zero Mapping Method”—in which one is faced with finding the DC gain of a system possessing an integration. The DC gain is that ratio of the steady-state output to the steady-state input, and a system possessing an integration has a steady-state output which is a ramp—there is no steady-state value.

1.1 Finding the DC Gain

Consider both continuous and discrete LTI systems:

\[
\frac{Y(s)}{U(s)} = G(s) \quad \text{or} \quad \frac{Y(z)}{U(z)} = G(z)
\]  

(1)

Finding the DC gain of an LTI system (either continuous or discrete) is based on the Final Value Theorem, and in either case involves the following formulæ:

Continuous System: \( \frac{y(\infty)}{u(\infty)} = G(s)_{s=0} = G(0) \)  
(2)

Discrete System: \( \frac{y(\infty)}{u(\infty)} = G(z)_{z=1} = G(1) \)  
(3)

In both cases, if the system has an integration (free \( s \) term or \( z^{-1} \) term in the denominator) the result will be \( \infty \).

1.2 One Solution—Differentiate!

There is a solution: find the ratio between the steady-state input and the steady-state derivative of the output. This requires one to differentiate the output. This is similar in both the continuous and discrete domains, but a little different...

1.2.1 Differentiation in the continuous domain

In the continuous, or “s” domain, differentiation is done by simply multiplying by \( s \), that is, referring to (1):

\[
\frac{\dot{Y}(s)}{U(s)} = sG(s)
\]

(4)

Please excuse the “bastard” notation of \( \dot{Y}(s) \)—this just represents the Laplace transform of \( \dot{y}(t) \).

1.2.2 Differentiation in the discrete domain

The derivative in the discrete domain can be done by a first difference:

\[
\hat{y}_k = \frac{y_k - y_{k-1}}{T} \quad \Rightarrow \quad \dot{Y}(z) = \frac{Y(z) - z^{-1}Y(z)}{T} = Y(z) \left[ 1 - \frac{z^{-1}}{T} \right] = Y(z) \left[ \frac{z - 1}{Tz} \right]
\]

(5)

Thus to differentiate in the discrete domain, simply multiply by \( \left[ \frac{z - 1}{Tz} \right] \).
2 An Example

2.1 Continuous System

Consider the continuous transfer function

\[ G(s) = \frac{5}{s(s+10)} \]  

To find the DC gain (steady-state gain) of this transfer function between the derivative of the output and the steady-state input, multiply by \( s \), then let \( s \to 0 \). Thus we have

\[ \text{“derivative” DC gain of } G(s) = \lim_{s \to 0} sG(s) = 0.5 \]  

2.2 Discrete System

Let’s use the “Pole-Zero Mapping Method” to find a \( G(z) \) as I tried in class. Use sampling period \( T = 0.02 \) seconds. Mapping both poles using \( z = e^{Ts} \) and placing two zeros at \( z = -1 \), we have

\[ G(z) = \frac{(z+1)^2}{(z-1)(z-0.8187)} \]  

But we need to make the DC gain of (8) match the DC gain of (6). We can still do this with the “derivative” gain, which we found in (7). Applying the result in (5) to this \( G(z) \), we get

\[ \text{“derivative” DC gain of } G(z) = \lim_{z \to 1} \left[ \frac{z-1}{T} \frac{(z+1)^2}{(z-1)(z-0.8187)} \right] = 1103.1 \]  

To make the DC gain of \( G(z) \) match that of \( G(s) \) we must apply a numerator constant which is equal to \( 0.5/1103.1 \). This results in final result of

\[ G(z)_{PZ} = \frac{(4.5325e-04)(z+1)^2}{(z-1)(z-0.8187)} \]  

2.3 Validation

I would encourage you to find the step response of both the \( G(s) \) and \( G(z) \) of (6) and (10) to verify that they are similar.