Chapter 7 HW 3 Solution

Problem 1. (a) Here we want to find the controlled system equations in terms of state vector

$$\begin{bmatrix} \mathbf{x} \\ \tilde{\mathbf{x}} \end{bmatrix} \tag{1}$$

First consider the state and output equations for the SISO plant:

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k) \tag{2}$$

$$y(k) = \mathbf{C}\mathbf{x}(k) \tag{3}$$

Next consider the state equation for the estimator, which is the same as for the plant plus the output error correction term:

$$\hat{\mathbf{x}}(k+1) = \mathbf{\Phi}\hat{\mathbf{x}}(k) + \mathbf{\Gamma}u(k) + \mathbf{L}[y(k) - \mathbf{C}\hat{\mathbf{x}}(k)] \tag{4}$$

Estimator error $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$, so if we substitute for $y(k)$ from (3) into (4), then subtract (4) from (2), we get

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k) \tag{5}$$

$$-(\hat{\mathbf{x}}(k+1) = \mathbf{\Phi}\hat{\mathbf{x}}(k) + \mathbf{\Gamma}u(k) + \mathbf{L}[\mathbf{C}\mathbf{x}(k) - \mathbf{C}\hat{\mathbf{x}}(k)]) \tag{6}$$

$$\tilde{\mathbf{x}}(k+1) = \mathbf{\Phi}\tilde{\mathbf{x}}(k) - \mathbf{L}\mathbf{C}\tilde{\mathbf{x}}(k) \tag{7}$$

which can be expressed as

$$\tilde{\mathbf{x}}(k+1) = [\mathbf{\Phi} - \mathbf{L}\mathbf{C}]\tilde{\mathbf{x}}(k) \tag{8}$$

Since there are not reference input terms in (8), the reference input $r(k)$ does not excite any estimator error.

Problem 2. For the DC motor + load system, introduce a reference input and simulate the response to a load motion of 10° in 0.5 seconds using a cubic polynomial trajectory.

Design of the control law and the estimator have been covered previously. I'm using a sampling frequency of 100 Hz ($T = 0.01$).

Control Gains. I selected "Butterworth" pole placement, which comes from filter design (a Butterworth filter as optimally flat frequency response over the passband). If there are n poles (3 in this case), one places $2n$ poles equally spaced around a circle of radius ω_n in the s-plane (the poles in the right-half plane are ignored). For this 3$^{\text{rd}}$ order system, the s-plane poles are at angles of ±120° and ±180°.

I selected a natural frequency of 15 Hz; this places the poles at

$$s = -94.2478, -47.1239 \pm j81.6210 \tag{9}$$

with corresponding z poles of

$$z = 0.3897, 0.4276 \pm j0.4548 \tag{10}$$

The control law gain matrix $\mathbf{K}$ is

$$\mathbf{K} = \begin{bmatrix} -0.6048 & 1.1229 & 0.0026774 \end{bmatrix} \tag{11}$$

Estimator Gains. I doubled the control law natural frequency (30 Hz), placed the three estimator poles likewise, and got

$$s = -188.4956, -94.248 \pm j163.24 \tag{12}$$

with discrete poles at

$$z = 0.1518, -0.0240 \pm j0.3889 \tag{13}$$

The estimator gain matrix $\mathbf{L}$ is

$$\mathbf{L} = \begin{bmatrix} -38.229 & 1.0819 & 146.36 \end{bmatrix}^T \tag{14}$$

Reference Input. The reference input is load position, or θ_l in degrees. Since that is also the measured system output, all "**C**" matrices are the same, that is

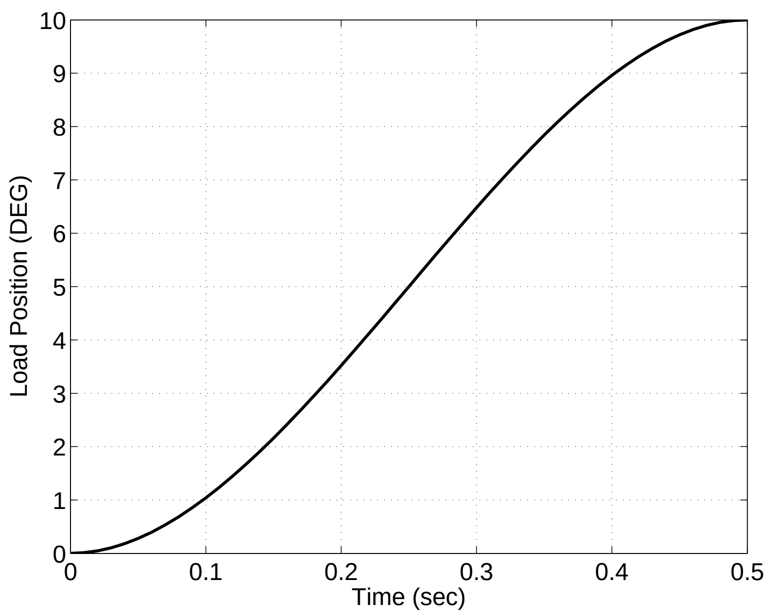
$$\mathbf{C} = \begin{bmatrix} 0 & 180/(n\pi) & 0 \end{bmatrix} \tag{15}$$

Using the methods of Section 7.8 (since the number of plant inputs and reference inputs is the same, we don't need the pseudoinverse), the reference input matrices are:

$$\mathbf{N}_x = \begin{bmatrix} 0 & 0.8727 & 0 \end{bmatrix}^T \tag{16}$$

$$N_u = 0 \text{ (makes sense since plant is Type 1)} \tag{17}$$

A plot of the 10° load motion in 0.5 seconds is shown below (by now I'm sure you're well acquainted with cubic polynomial trajectories).



System Simulation. I chose to perform a Simulink-based simulation using the original continuous plant (I actually used the continuous transfer function $G(s)$ of the plant).

This transfer function (obtained using `ss2tf`) is

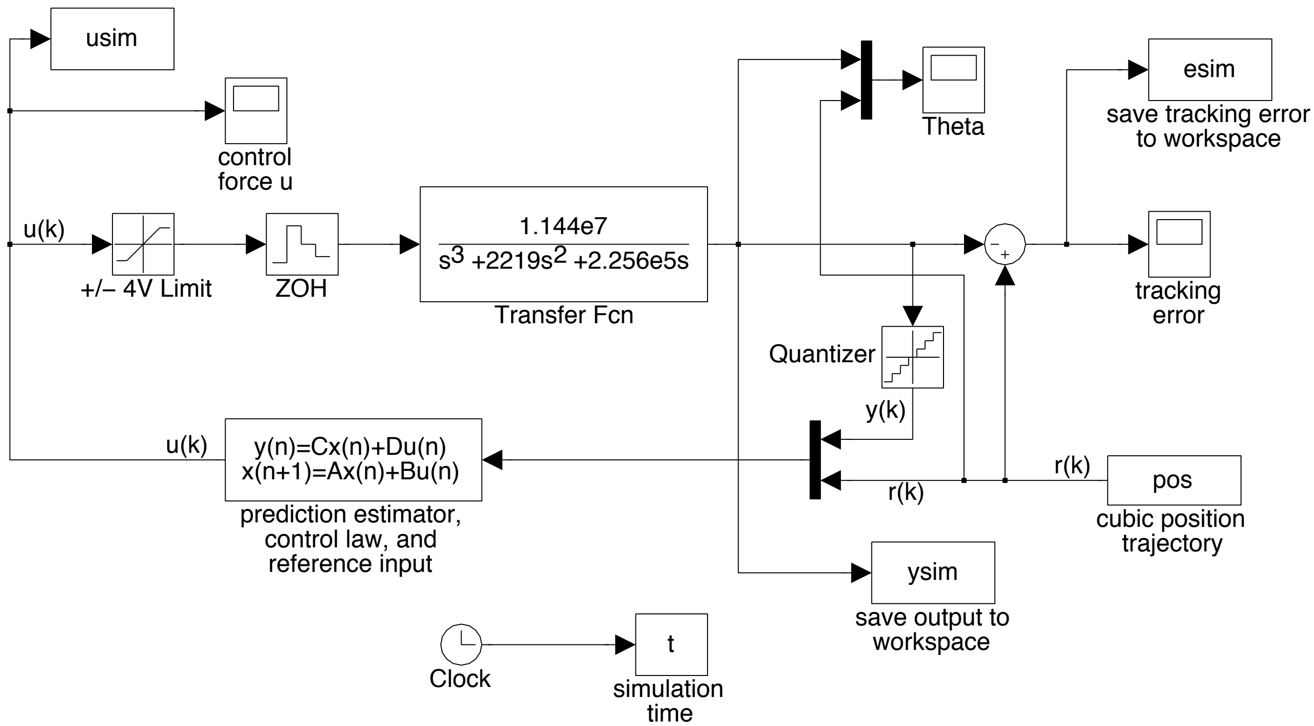
$$G(s) = \frac{(-4.547e-13)s^2 + (2.91e-11)s + 1.144e07}{s(s^2 + 2219s + 2.256e05)} \tag{18}$$

Note the extremely small numerator coefficients. When those are neglected, one is left with

$$G(s) = \frac{\theta_l(s)}{V(s)} = \frac{1.144e07}{s(s^2 + 2219s + 2.256e05)} \quad \frac{\text{deg}}{\text{V}} \tag{19}$$

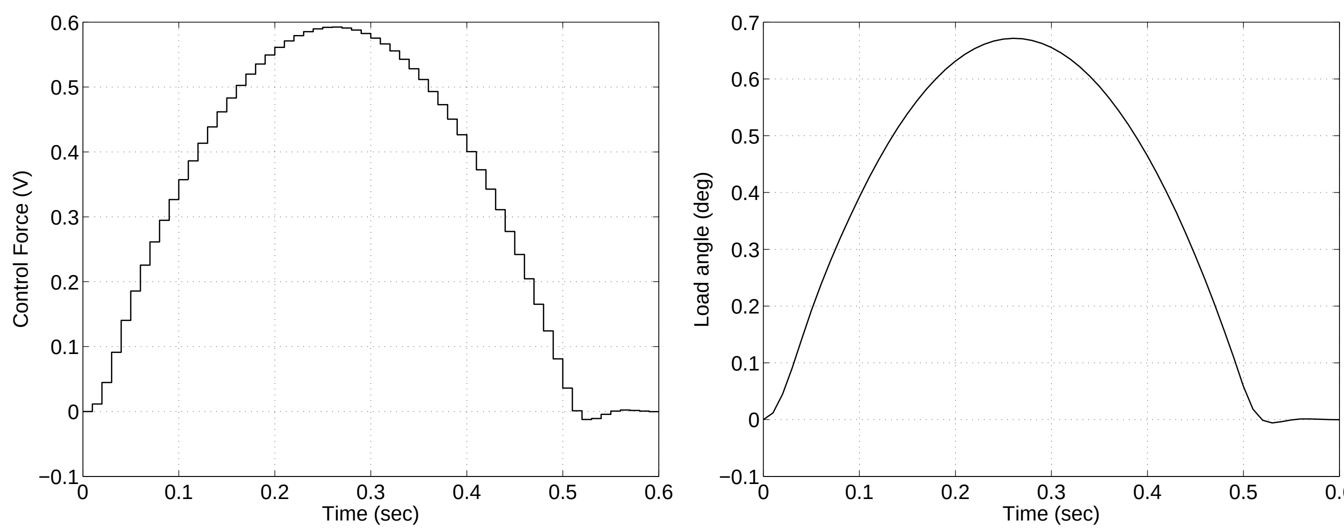
This is the transfer function I used in the Simulink model.

Simulink Model. A figure showing my Simulink model is below.



This is virtually identical to the state-space project simulation. I'll show some runs of this model during class. Note that the "Clock" block generates the actual time vector used in the simulation (not spaced at T due to the continuous block $G(s)$ and thereby-needed variable-step integrator).

Response Plots. The relevant response plots are probably control force u, and tracking error e; those are shown below.



(a) Control Force. (b) Tracking Error.

Figure 1: Response Plots for Motor & Load.

This tracking behavior is *not too great!* This is because state-space control doesn't inherently introduce any derivative action! The key to improving performance is adding a derivative input (*i.e.* MIMO system).