# Chapter 7 HW 1 Solution

**Problem 1.** First compute the $\mathbf{\Psi}$ matrix, which is given by series expansion

$$\mathbf{\Psi} = \mathbf{I} + \frac{\mathbf{A}T}{2!} + \frac{\mathbf{A}^2T^2}{3!} + \cdots, \tag{1}$$

and when $\mathbf{A}$ and $\mathbf{B}$ and $T$ are substituted we obtain

$$\mathbf{\Psi} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & 0.05\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} + \cdots\text{[all zeros]}\cdots = \begin{bmatrix}1 & 0.05\\ 0 & 1\end{bmatrix} \tag{2}$$

Using $\mathbf{\Psi}$ in (7.20) and (7.21) yields

$$\boxed{\mathbf{\Phi} = \begin{bmatrix}1 & 0.1\\ 0 & 1\end{bmatrix}, \qquad \mathbf{\Gamma} = \begin{bmatrix}0.005\\ 0.1\end{bmatrix}} \tag{3}$$

which agrees with the MATLAB `c2d` function (and my notes). Note that the convergence of matrix $\mathbf{\Psi}$ was quite rapid. It's not usually quite *that* fast.

**Problem 2.** We have the armature voltage-controlled DC motor driving a load inertia *via* a gear train.

As before, the state vector is $\mathbf{x} = \begin{bmatrix}i_a & \theta_m & \omega_m\end{bmatrix}^T = \begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}^T$, and the input is applied voltage $e_a = u$. We are certainly interested in load position (deg), we might also be interested in load velocity (deg/s), and motor current (A) as outputs (for plotting purposes; the *real* output is probably load position only), and the feedthrough matrix $\mathbf{D}$ is zero. Thus the output is $\mathbf{y} = \begin{bmatrix}i_a(\text{A}) & \theta_l(\text{deg}) & \omega_l(\text{deg/s})\end{bmatrix}^T$, and we have

$$\mathbf{A} = \begin{bmatrix}-R_a/L_a & 0 & -K_b/L_a\\ 0 & 0 & 1\\ K_t/J_t & 0 & -B_t/J_t\end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix}1/L_a\\ 0\\ 0\end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix}1 & 0 & 0\\ 0 & 180/(n\pi) & 0\\ 0 & 0 & 180/(n\pi)\end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}. \tag{4}$$

The inertia and damping terms $J_t$ and $B_t$ represent the "total" inertia of motor armature plus load (remember that the load inertia/damping are *referred* to the motor shaft by dividing by gear ratio $n^2$).

This motor/load is *uncontrolled*, that is, there is no feedback. There is no way to control the motor's (or load's) position, *etc.* If you give the motor/load some non-zero initial state it will just do whatever—it certainly won't return to zero!

If we design a *state-feedback controller* for this motor/load we can impose closed-loop control on the entire state vector; the motor/load system will then have "good" natural behavior. Later we can think about adding a reference input to control the particular variable of interest (probably load position or velocity).

To design a state-feedback controller, we need to place the poles of the controlled system at reasonable locations. Since the velocity step response had a time constant of near 0.01 second (which corresponds to a pole at $s = -100$), it is reasonable to place all three system poles somewhere near $s = -100$. I decided to place one pole at -100, and the other two as complex conjugates at a radius of 100 and angles of 45° from the negative real axis.

Thus the desired $s$-plane poles are:

$$s = \underbrace{-100}_{\tau=0.01}, \ \underbrace{-70 \pm j70}_{\zeta=0.7,\omega_n=100} \tag{5}$$

Mapping these poles to the $z$-plane yields

$$z = \begin{bmatrix}0.3679\\ 0.3798 + j0.3199\\ 0.3798 - j0.3199\end{bmatrix} \tag{6}$$

where there is really no reason to keep four decimal places of accuracy!

Before we go further, we need to discretize the plant model; using MATLAB `c2d` the resulting system and input matrices are:

$$\mathbf{\Phi} = \begin{bmatrix} -0.0181 & 0 & -0.0055 \\ 0.0195 & 1.0000 & 0.0064 \\ 1.1806 & 0 & 0.3616 \end{bmatrix}, \qquad \mathbf{\Gamma} = \begin{bmatrix} 0.2538 \\ 0.1573 \\ 28.2421 \end{bmatrix} \tag{7}$$

We can then use MATLAB `place` to find the state feedback gain matrix $\mathbf{K}$. The result is

```
>> K = place(Phi,Gamma,z_poles)

K = -0.3512    1.0596    0.0049
```

Since the feedback law is $u = -\mathbf{K}\mathbf{x}$, the control law is actually

$$u = 0.35 i_a - 1.06\theta_m - 0.005\omega_m, \tag{8}$$

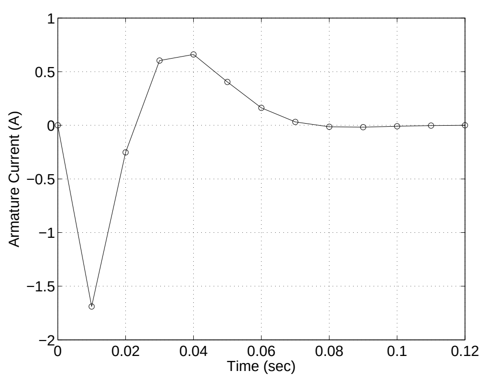
which represents *positive* current feedback, and negative position and velocity feedback.

**Response to initial condition.** To confirm the behavior of the controlled plant, let's take as an initial state simply a motor "windup" of one revolution (note that this is 1/50 revolution for the load, or about 7°). Armature current and motor velocity will be left at zero. Thus $\mathbf{x}_0 = \begin{bmatrix} 0 & 2\pi & 0 \end{bmatrix}^T$. Because of the choice of our $\mathbf{C}$ matrix the output will be armature current, and load position and velocity in degrees.

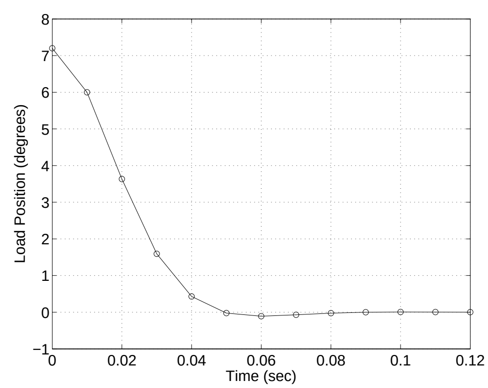
I created the closed-loop system with a zero input matrix, since we don't really have an input yet. And $\mathbf{D}$ is still zero.

```
>> clsys_d = ss(Phi-Gamma*K,zeros(3,1),C,D,T);
```

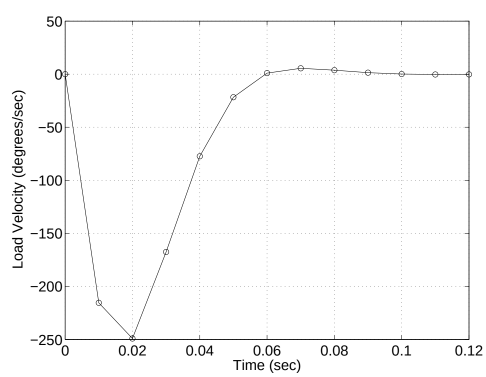
The plots of the three outputs are shown in Figure 1 below:




(a) Current response.




(b) Load position response.




(c) Load velocity response.

Figure 1: Motor/load response to 7° load initial condition.