Chapter 6 HW Solution

Problem 1. We have an armature voltage-controlled DC motor driving a load inertia *via* a gear train. All necessary numerical parameters are given.

(a) The state vector is $\mathbf{x} = \begin{bmatrix} i_a & \theta_m & \omega_m \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$. The input is applied voltage $e_a = u$. Here we are analyzing the motor alone; the load is not yet connected.

The state equations for the motor are:

$$\frac{di_a}{dt} = -\frac{R_a}{L_a} i_a - \frac{K_b}{L_a}\omega_m + \frac{1}{L_a} e_a \tag{1}$$

$$\frac{d\theta_m}{dt} = \omega_m \tag{2}$$

$$\frac{d\omega_m}{dt} = -\frac{B_m}{J_m}\omega_m + \frac{K_t}{J_m} i_a \tag{3}$$

The $\mathbf{A}$ and $\mathbf{B}$ matrices are

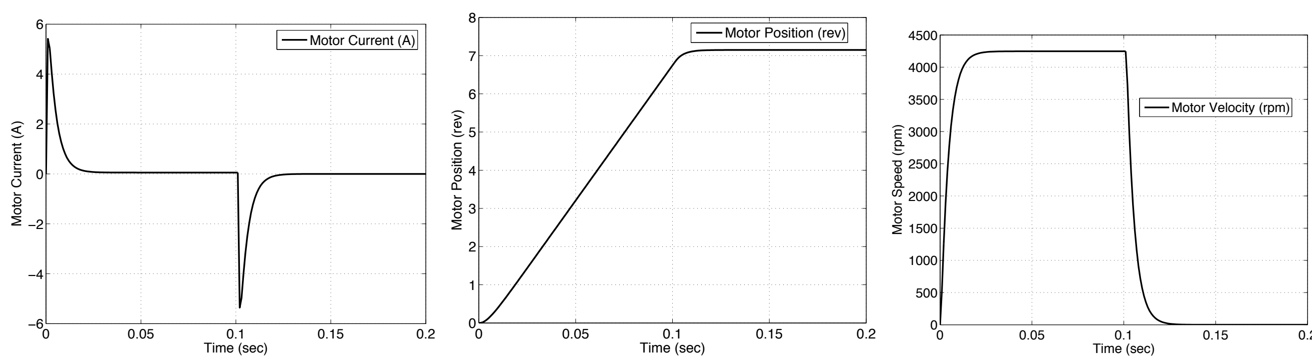
$$\mathbf{A} = \begin{bmatrix} -R_a/L_a & 0 & -K_b/L_a \\ 0 & 0 & 1 \\ K_t/J_m & 0 & -B_m/J_m \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

You should use care when substituting numerical parameters to make sure the units are consistent.

Since we are interested in plotting current i_a (A), motor displacement θ_m (revolutions), and motor speed ω_m (rpm), I selected the $\mathbf{C}$ matrix to be

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2\pi & 0 \\ 0 & 0 & 60/2\pi \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \tag{5}$$

The responses to the 10V pulse of length 0.1 second are shown below.



(a) Armature current (A). (b) Motor shaft position (revolutions). (c) Motor shaft speed (rpm).

Figure 1: Motor responses to pulse input.

(b) Now you're adding the load dynamics to the motor. As stated in the problem, when a motor drives a load through a gear train with ratio n ($n > 1$ for speed reduction) the load inertia "referred" to the motor shaft (the usual practice) is the load inertia divided by n^2. I kept the same state vector as in part (a), but modified the output matrix $\mathbf{C}$ to produce the desired output.

The inertia and damping of the load are

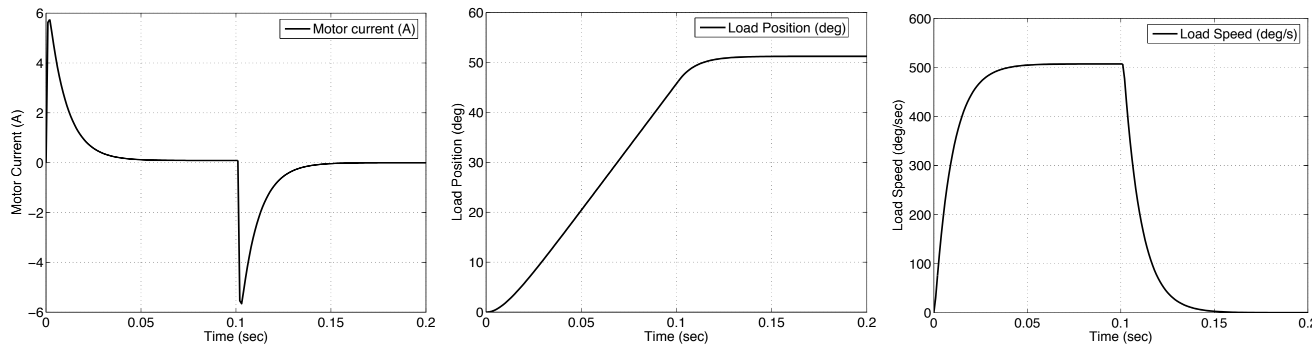
$$J_l = 0.00424 \text{ kg-m}^2 \tag{6}$$

$$B_l = 0.004 \text{ N-m-s} \tag{7}$$

Adding the load inertia and damping roughly doubles the motor inertia and damping. To get load position and velocity in degrees requires a **C** matrix of

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 180/n\pi & 0 \\ 0 & 0 & 180/n\pi \end{bmatrix} \tag{8}$$

Plots of the load response to the pulse are shown below.



(a) Armature current (A). (b) Load shaft position (degrees). (c) Load shaft speed (degrees/sec).

Figure 2: Load responses to pulse input.

(c) Now ignore the armature inductance L_a; the armature voltage equation becomes

$$e_a - i_a R_a - K_b \omega_m = 0. \tag{9}$$

and the state vector is $\mathbf{x} = \begin{bmatrix} \theta_m & \omega_m \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$. Current $\imath_z$ is no longer a state variable, and can be eliminated, resulting in

$$J_m \dot{\omega}_m + \underbrace{\left(B_m + \frac{K_t K_b}{R_a} \right)}_{\text{effective damping } B_{eq}} \omega_m = \frac{K_t}{R_a} e_a \tag{10}$$

Note that the *back emf* (K_b) manifests itself as an "electronic damping" term (also true when L_a considered; just more obvious here). It will simplify the analysis if we define an "equivalent damping" term B_{eq} as:

$$B_{eq} \triangleq B_m + \frac{K_t K_b}{R_a} \tag{11}$$

The state equations are now

$$\dot{\theta}_m = \omega_m \tag{12}$$

$$\dot{\omega}_m = -\frac{B_{eq}}{J_m}\omega_m + \frac{K_t}{J_m R_a} e_a \tag{13}$$

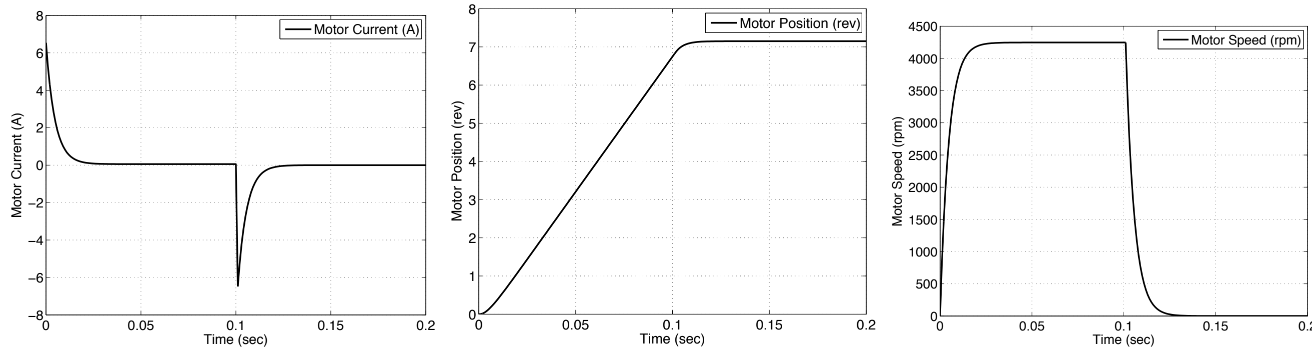
Now the motor model is 2^{nd} order, and the $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ matrices are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{B_{eq}}{J_m} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{K_t}{J_m R_a} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1/2\pi & 0 \\ 0 & 60/2\pi \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{14}$$

The motor position and velocity results are directly available from the simulation, but armature current i_a must be computed using (9)—of course you must have ω_m in rad/s, not rpm!

$$i_a = \frac{e_a - K_b \omega_m}{R_a}. \tag{15}$$

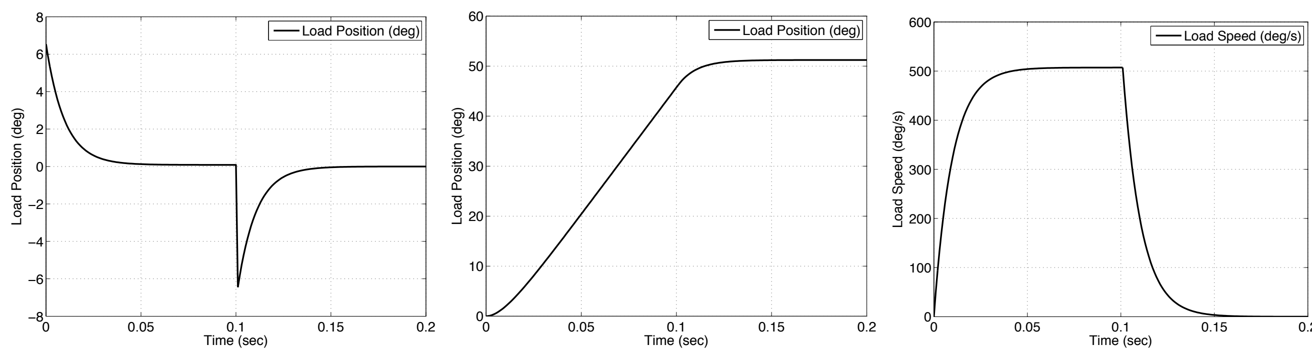
The same three plots as before are shown in Figure 3 (this is the motor alone). They are virtually identical to Figure 1, although if you look closely at the current plot in Figure 1(a) you can see a slight transient as the current initially rises. In Figure 3(a) the current rises immediately—there are no "current dynamics." This would be easier to see if I expanded the current plot, but I'll leave that for you to do.



(a) Armature current (A). (b) Motor shaft position (degrees). (c) Motor shaft speed (degrees/sec).

Figure 3: Motor responses to pulse input; armature inductance L_a neglected.

Finally, repeat the above procedure but add the load. The effective inertia and damping increase as before, but the modeling is the same. Plots are shown in Figure 4.



(a) Armature current (A). (b) Motor shaft position (degrees). (c) Motor shaft speed (degrees/sec).

Figure 4: Load responses to pulse input; armature inductance L_a neglected.

Again, the responses are almost identical. Neglecting armature inductance is commonly done, and you can see why...

Problem 2. For some reason I'm getting strange results—perhaps a simple mistake, but I didn't have time to find my error. If you do this problem (with reasonable results), give yourself 3 extra points.