Chapter 5 HW 2 Solution

Problem 1. This is an unstable plant whose linear transfer results from the linearization of the "inverse-square" magnetic law.

(a) The numerical transfer function from i to x is

$$G(s) = \frac{X(s)}{I(s)} = \frac{20}{s^2 - 1000} = \frac{20}{s \pm 32} \quad (1)$$

We have not yet specified an input to this system; this does not pose a problem in using root locus because the closed-loop system will have characteristic equation $1 + KG(s) = 0$, which is excatly what is needed for the root locus.

The s-plane root locus will branches emanating from the poles at -32 and +32 coming together at the origin, then proceeding up and down along the $j\omega$ axes. So the system is marginally stable at best.

(b) Using sample period $T = 0.02$, the discrete transfer function is

$$G(z) = \frac{0.004135z + 0.004135}{z^2 - 2.414z + 1} = \frac{0.004135(z+1)}{(z - 0.5313)(z - 1.8822)} \quad (2)$$

so the $G(z)$ is also unstable as expected.

(c) The performance specifications yield:

- PO $\leq 20\% \implies \zeta \geq 0.45$
- $t_r \leq 0.1 \implies \omega_n \geq \frac{1.8}{0.1} = 18$ rad/s
- 2% $t_s = \frac{3.9}{\zeta\omega_n} \implies \zeta\omega_n \geq 10 \implies r = e^{\zeta\omega_n T} \leq 0.85$

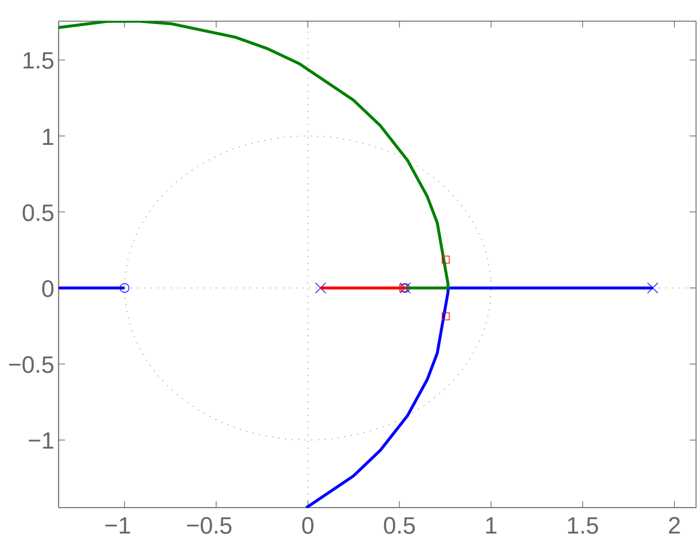
I selected $\zeta = 0.7$ and $\omega_n = 18$ rad/s as desired performance criteria; this also satisfied the t_s specification. The corresponding desired poles are:

$$s = -12.6000 + j12.8546 \implies z = 0.7517 + j0.1976 \quad (3)$$

With a lead compensator $D(z)$ whose zero cancels the plant pole at $z = 0.53$, I used the root-locus angle condition to find the pole location of $D(z$, whose resulting transfer function is

$$D(z) = \frac{z - 0.53}{z - 0.07} \quad (4)$$

The root-locus diagram is shown below, with squares at the desired pole location ($K = 111$).



The resulting closed-loop characteristic polynomial is

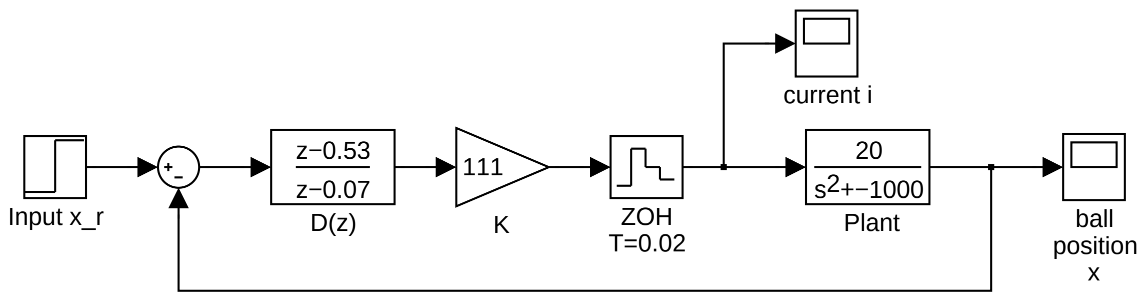
$$z^3 - 2.025z^2 + 1.385z - 0.3133 = (z - 0.7519 \pm j0.1905)(z - 0.5207) \tag{5}$$

and the dominant closed-loop poles are very close to the desired locations.

(d) Finally introduce reference input x_r in the "standard" manner, and find the two desired step responses. Using lead compensator $D(z)$ from (c) I was able to achieve the following response characteristics:

$$t_r = 0.12 \text{ sec} \tag{6}$$
$$t_s\ (2\%) = 0.34 \text{ sec} \tag{7}$$
$$PO = 4\% \text{ overshoot} \tag{8}$$

I decided to use `Simulink` to simulate this system, since I could examine both the ball position x and the amplifier current i using the same model.



From the two "scopes" I found that a step input of x_r = 1 mm = 0.001 m caused a peak x of nearly 10 mm and a peak current of about 0.5 A (this will be shown in class). Since maintaining static equilibrium requires 0.49 A, only 0.51 A is "left" for control. Thus a maximum current of about 0.5A causes a ball position of about 1 cm. Since the sensor range is only 1/4 cm the sensor is the limiting factor and the range for control is $\pm 1/4$ cm.

Problem 2. For the "cruise control" problem:

(a) The performance specifications are:

- $t_r = 5 \text{ sec} \Longrightarrow \omega_n = \frac{1.8}{5} = 0.36$ rad/s
- $\zeta = 1 \Longrightarrow$ two dominant poles at $s = -\zeta\omega_n = -0.36 \Longrightarrow z = 0.83$

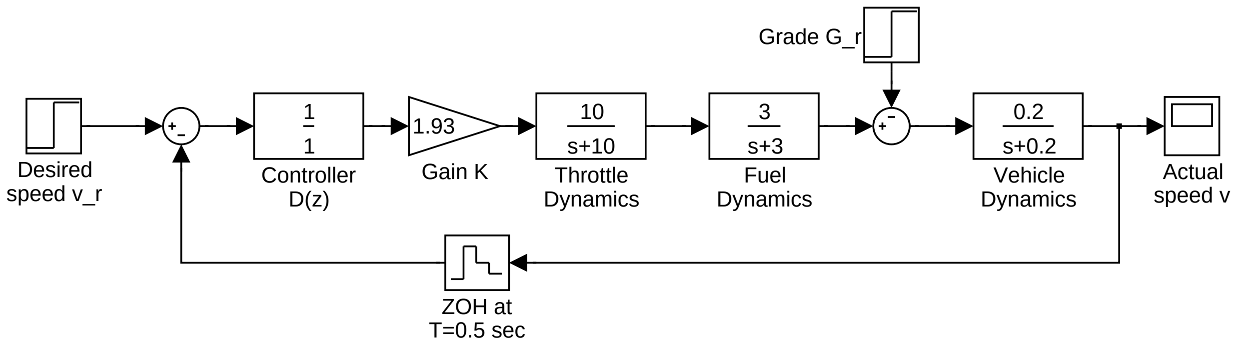
When we discretize the forward path at $T = 0.5$ we obtain

$$G(z) = \frac{0.03346z^2 + 0.03855z + 0.001427}{z^3 - 1.135z^2 + 0.2095z - 0.00136} = \frac{0.03346(z + 1.1139)(z + 0.0383)}{(z - 0.9048)(z - 0.2231)(z - 0.0067)} \tag{9}$$

With proportional control $D(z) = K$ the root locus shows that critical damping occurs when $K = 1.93$; however the natural frequency of these dominant poles will be 1.24 rad/s—much higher than we need!

The unit step response is plenty fast enough ($t_r \approx 3$ sec) but it is noteworthy that the output only reaches a steady-state value of 0.67—there is a large steady-state error. This doesn't sound like the behavior we want from a cruise control, which will be confirmed in part (b).

(b) I constructed a `Simulink` block diagram of the system, since it's fairly easy to introduce multiple inputs (and display multiple outputs). This block diagram with $D(z) = 1$ and $K = 1.93$ is shown on the next page.



When I applied a 3% grade disturbance $G_r = 3$ to the system (desired $v_r = 0$) the resulting speed change was a little over 1 mph (I changed G_r to enter with a "-" sign; more indicative of the real effect of a hill). So the vehicle would slow down when encountering a grade. This is not supposed to happen (within the limits of engine torque). But this is a Type 0 system—we need to increase Type.

(c) To reduce the error to zero, you need to increase the system *Type*. This is done by adding an integrator, so we'll try PI control. A PI controller in the z-domain is of the form

$$D(z) = K_P + K_I \frac{z}{z-1} = \frac{K_P(z-1) + K_I z}{z-1} = \frac{(K_P + K_I)\,z - K_P}{z-1} = (K_P + K_I)\,\frac{z - \frac{K_P}{K_P+K_I}}{z-1}$$
$$= K\frac{z-a}{z-1}$$

where $0 < a < 1$.

I used pole-zero cancellation; it made matters a little easier, thus

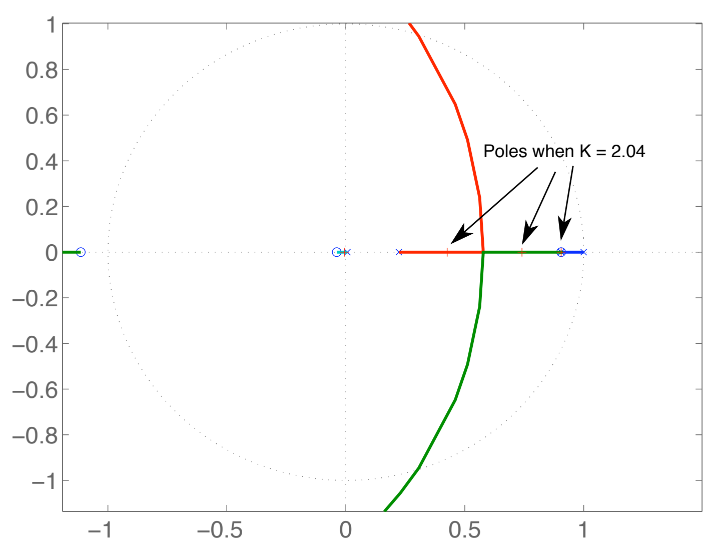
$$D(z) = \frac{z - 0.9048}{z-1} \tag{10}$$

With the pole-zero cancellation of $D(z)$, it turns out that we will not have two dominant *complex* poles, but rather a *single real* dominant pole. So this changes the strategy a little on finding desired pole location.

A real pole has a *time constant*, and we usually figure that the complete response occurs in about 3 time constants, thus for a 5 second rise time:

$$3\tau = 5 \implies \tau = \frac{5}{3} \implies s = -\frac{3}{5} = -0.6 \implies z = e^{sT} = 0.74 \tag{11}$$

The root locus diagram of the PI controlled system is shown below.



The value of K that places the real pole at 0.74 is: $K = 2.04$. The `Simulink` diagram of the system with this $D(z)$ and K is shown below. The responses of this system will be shown in class. Both the rise time and disturbance behavior are satisfactory, although speed recover due to road grade takes about 20 seconds; this should probably be faster.