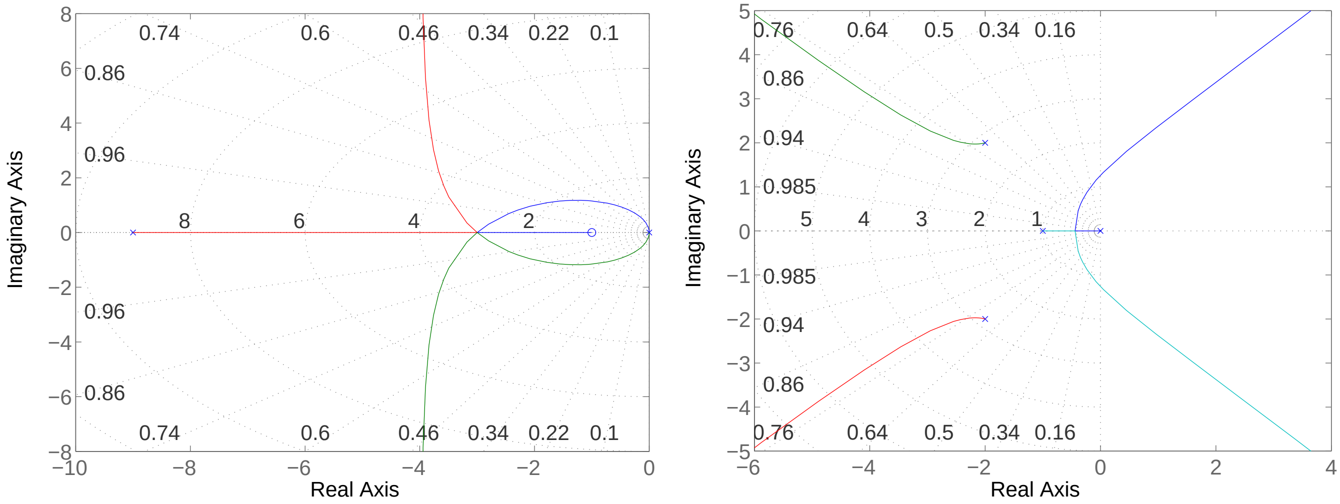
Chapter 5 HW Solution

Problem 1. These three characteristic equations were already in "root-locus" form. They can easily be restored to "polynomial" form, *e.g.*

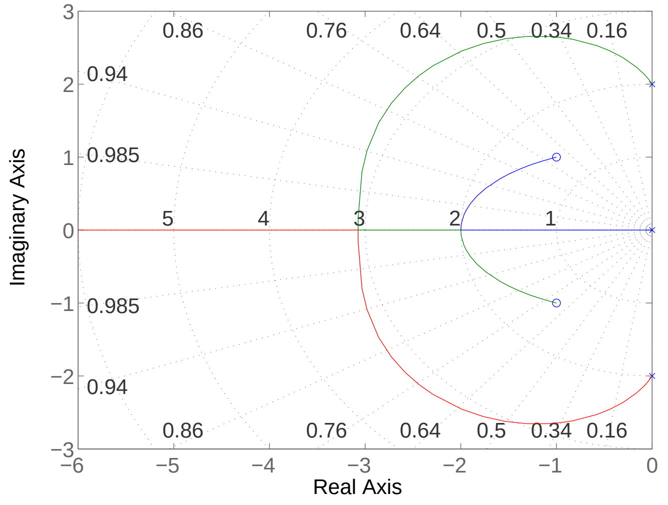
$$1 + K\frac{s+1}{s^2(s+9)} = 0 \qquad \Longrightarrow \qquad s^2(s+9) + K(s+1) = 0 \tag{1}$$

The three s-plane root locus plots as drawn by MATLAB are shown below.



(a) Double pole at origin.

(b) Complex poles and real poles.



(c) Complex zeros and poles.

Figure 1: Root-locus diagrams for Problem 1.

Although we used MATLAB for doing these, I think you should know some of the "sketching rules" such as: (1) real-axis branches, (2) asymptotes (their angles and intersection point), (3) rough approximations of the angle and magnitude conditions. Then you can recognize if the computer is giving you correct information. Remember: what you are creating are parametric plots of all possible closed-loop pole locations.

Problem 2. Here we are trying to control the orientation of a pure inertia using torque as the actuating variable. A block diagram of the system is shown in Figure 2 below. Variable $\tau(s)$ is the torque applied to the inertia (not needed for this problem—I just added it for illustrative purposes).

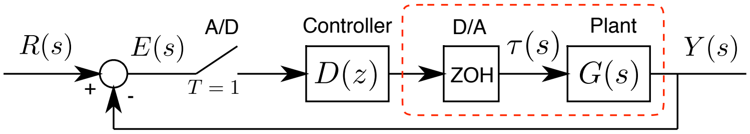


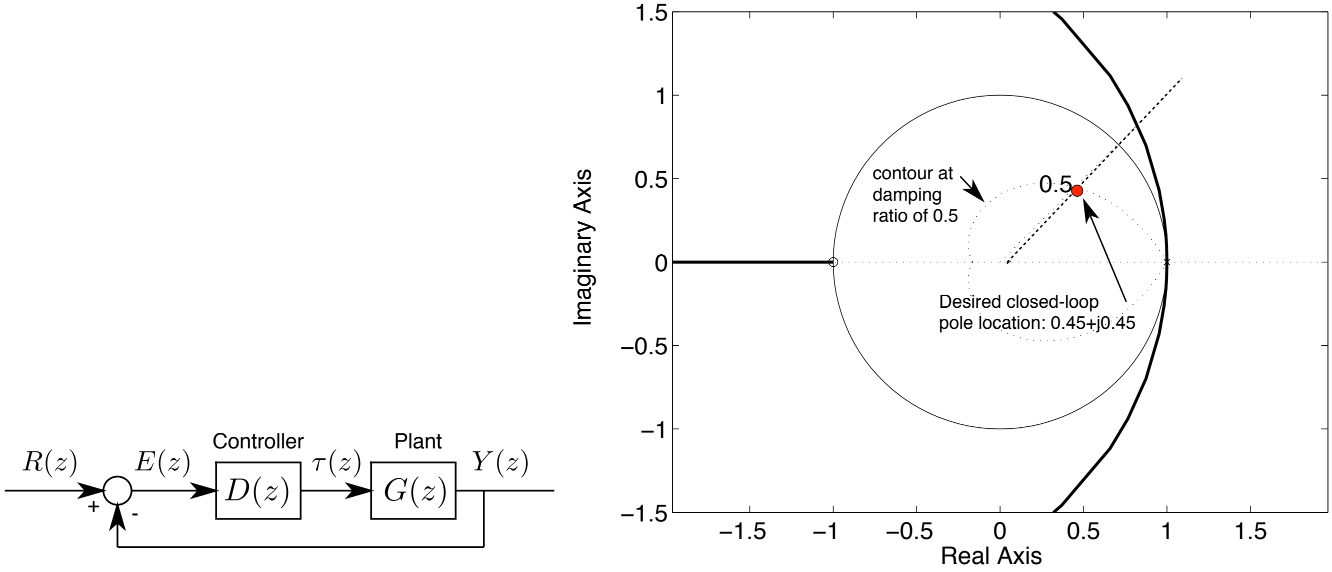
Figure 2: Block diagram of system.

(a) The portion of Figure 2 within the red dashed oval is what constitutes $G(z)$. Since the plant $G(s)$ is preceded by a ZOH you should use the `'zoh'` method in MATLAB `c2d`. Using a sample period of $T = 1$ second, the result is

$$\boxed{G(z) = \frac{0.5(z+1)}{(z-1)^2} = \frac{Y(z)}{\tau(z)}} \tag{2}$$

Note that this discrete plant transfer function relates samples of the output $Y(z)$ to samples of the torque input $\tau(z)$.

(b) A block diagram of the discretized system is shown below in Figure 3(a).



(a) Discretized block diagram.

(b) Root-locus using proportional control.

Figure 3: Discretized block diagram and root-locus using proportional control.

Using proportional control, $D(z) = K$, the closed-loop transfer function, characteristic equation, and corresponding root-locus form of the characteristic equation is:

$$\frac{Y(z)}{R(z)} = \frac{0.5K(z+1)}{(z-1)^2 + 0.5K(z+1)} \quad \Longrightarrow \quad (z-1)^2 + 0.5K(z+1) = 0 \quad \Longrightarrow \quad 1 + K\frac{0.5(z+1)}{(z-1)^2} = 0 \tag{3}$$

The root-locus diagram of (3) is shown in Figure 3(b). The branches are always outside the unit circle, hence the system is always unstable. Proportional control is unsatisfactory.

(c) We have a specification of $\zeta = 0.5$ and $\theta = 45°$ for the dominant closed-loop poles. The angle of $\theta = 45°$ means the poles are along a radial line emanating outward from the z-plane origin at an angle of 45°. Hence the z-plane pole location is of the form $z = \sigma + j\sigma$.

To find the desired z-plane pole location you can draw a radial line at 45° from the origin and see where it intersects the $\zeta = 0.5$ line in Figure 5.6 in the notes, or use the MATLAB `zgrid` function with $\zeta = 0.5$ and ω_n = anything (note that `zgrid` often doesn't draw the z-plane with equal scaling; you can execute an `axis equal` command to fix that (I think I do this in class sometimes).

However you do it, I found the desired z-plane pole location to be around $z = 0.45 + j0.45$; this location is shown as the red circle in Figure 3(b).

As in class the form of the *lead compensator* used for $D(z)$ is

$$D(z) = K\frac{z+b}{z+a}, \quad b < a \tag{4}$$

My approach with the lead compensator is to consider the s-plane, and place the lead zero at about 1/3 of the distance from the origin as the real part of the desired closed-loop pole location. This requires converting our desired z-plane pole location to the s-plane:

$$z = 0.45 + j0.45, \text{ but since } s = \frac{1}{T}\ln z \text{ we have } s = -0.452 + j0.785. \tag{5}$$

Thus I will place the lead zero at $s = -0.150 \implies z = 0.86$.

Figure 4 shows the root-locus angle condition applied to the desired pole location; the vectors are all drawn from the poles and zeros to this desired pole location.

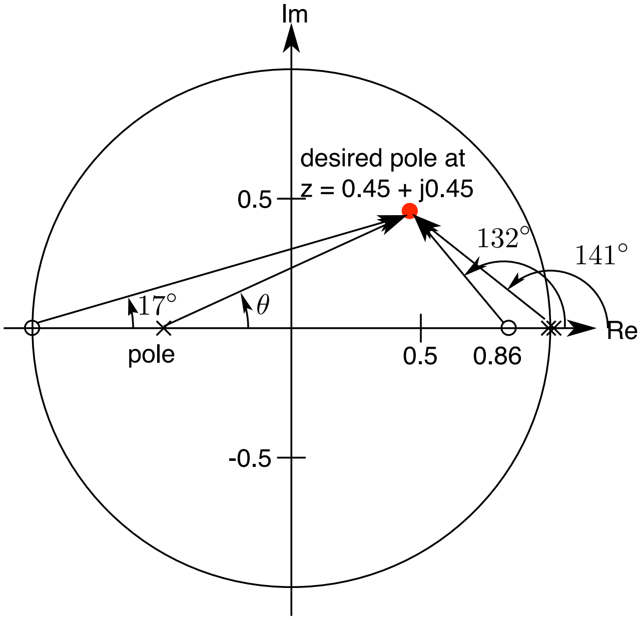


Figure 4: Angle condition applied to this compensated system.

The angle condition requires that the sum of the angles from poles to the desired pole minus the sum of the angles from zeros to the desired pole must equal ±180°. For the situation in Figure 4 we have

$$2(141°) + \theta - 132° - 17° = \pm 180° \tag{6}$$

where angle θ is the angle of the vector from the lead compensator pole to the desired closed-loop pole. Thus we find that $\theta = 47°$, and the compensator pole is at 0.03. Therefore

$$D(z) = K\frac{z-0.86}{z-0.03} \tag{7}$$

and the final step is to find K. This is done using the root-locus *magnitude condition*, which is conveniently done using the MATLAB function `rlocfind`. The new form of the closed-loop transfer function is

$$\frac{Y(z)}{R(z)} = \frac{0.5K(z-0.86)(z+1)}{(z-1)^2(z-0.03)+0.5K(z-0.86)(z+1)} \tag{8}$$

and the new characteristic equation is

$$(z-1)^2(z-0.03)+0.5K(z-0.86)(z+1)=0 \tag{9}$$

which has a root-locus form

$$1+K\frac{0.5(z-0.86)(z+1)}{(z-1)^2(z-0.03)}=0 \tag{10}$$

The root locus diagram of (9) is shown in Figure 5, and we see that the locus goes through the desired points! The closed-loop pole locations are shown by red crosses, and the value of K that places the dominant closed-loop poles where we want them is $K = 0.677$. Note that there is a closed-loop pole quite near the "+1" point, but it is also near a zero, which tends to reduce its effect. So the complex poles may still be dominant.

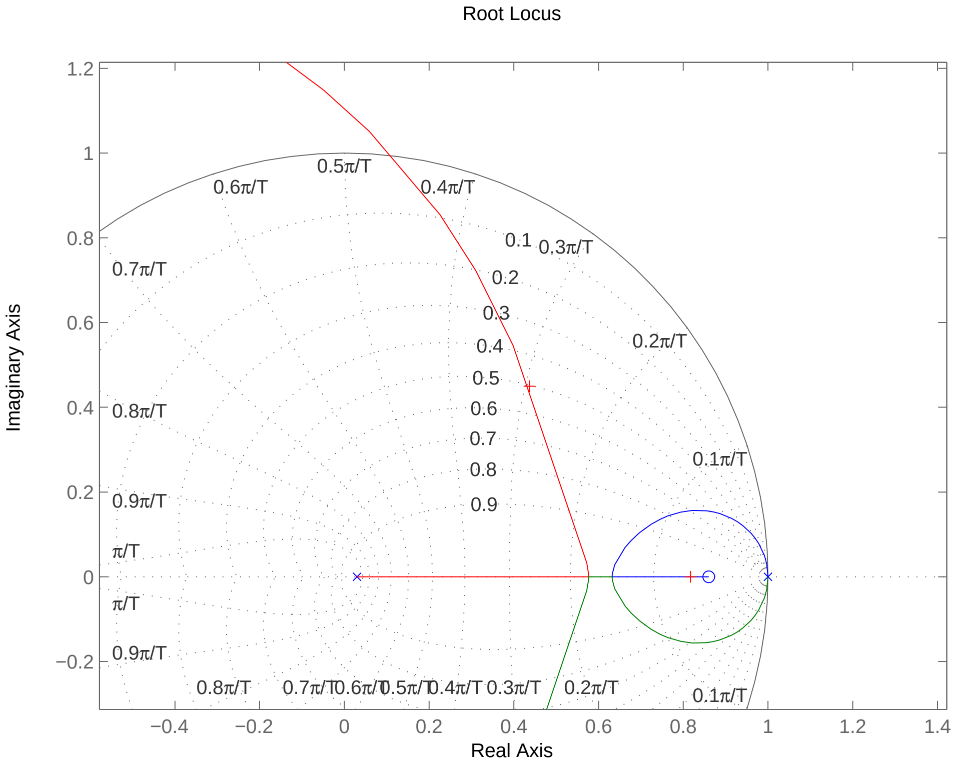


Figure 5: Root locus of compensated system.

Using the MATLAB `sys = feedback(frwrd_TF,fdbk_TF)` function to find the closed-loop transfer function, we get

$$\frac{Y(z)}{R(z)} = \frac{0.3387z^2+0.04741z-0.2913}{z^3-1.691z^2+1.107z-0.3213} \tag{11}$$

which can be factored to yield

$$\frac{Y(z)}{R(z)} = \frac{0.33866(z+1)(z-0.86)}{(z-0.4370\pm j0.4495)(z-0.8173)} \tag{12}$$

So after all this, the dominant poles are pretty much where we want them. Also note that the zero of $D(z)$ does show up as a closed-loop zeros (the one at $z = 0.86$). This will give the system ability to respond quickly to time-varying inputs. The unit step response is on the next page.

The unit step response is shown below. I used the `[y,t] = step(sys)` function to get the step response, then plotted it myself. Recall that the response between samples is the plant's ($G(s)$) step response (which is parabolic). But a straight line is close enough.

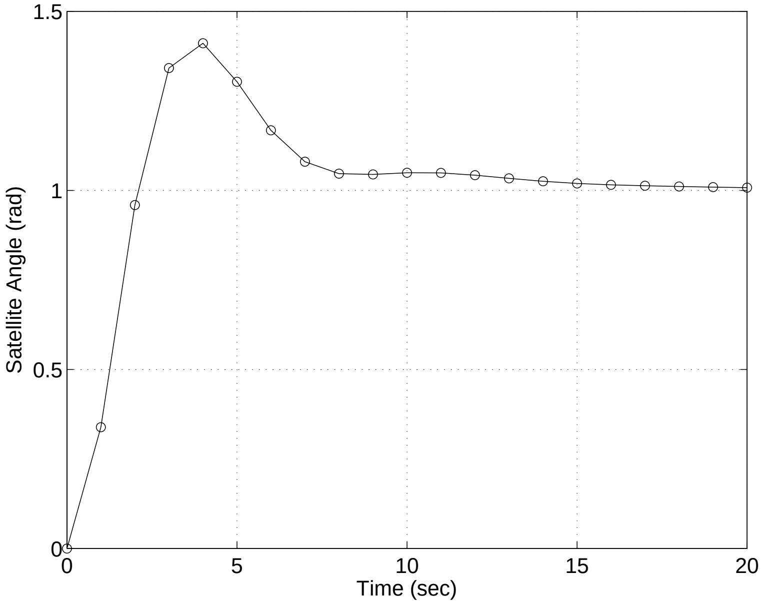


Figure 6: Unit step response of compensated system.

Probably the most surprising thing about Figure 6 is the large overshoot (41%), which is not expected of a system with a damping ratio $\zeta = 0.5$. Normally we would expect around 16% overshoot when $\zeta = 0.5$.

This large overshoot is due to the closed-loop zero at $z = 0.86$, which tends to speed up the response (it's a "pseudo-differentiator"). This is actually a **good thing**, which you will see when we confront the project. Many systems will never be subjected to a step input, which can be quite harsh, especially at large amplitudes. The fast response shown in Figure 6 is just what is needed to track a time-varying reference input.

A better input to use to illustrate the behavior of this system would be some kind of smoothly-varying reference input; we will use such an input during the projects.

Finally, note that the response of Figure 6 doesn't immediately go to its final value of 1.00, but asymptotically approaches it over a period of 10-15 seconds. This is the effect of the *third* closed-loop pole at $z = 0.8173$ (see equation (12)). The time constant τ of this pole can be found by mapping it back to the z-plane:

$$s = \frac{1}{T}\ln(z) = -0.2017 \quad \Longrightarrow \quad \tau = \frac{1}{|-0.2017|} \approx 5 \text{ sec}$$

This relatively slow time constant is responsible for the slow asymptotic final "approach" of the step response.

(d) It is illustrative to look at the Bode plots (*vs* Hz) of both the open- and closed-loop transfer functions. Figure 7 shows the open-loop $KD(z)G(z)$: The phase margin is nearly 40° which implies pretty good stability. The gain margin is around 9 dB; also good. The open-loop gain crossover frequency is about 0.1 Hz; although bandwidth was not specified, the closed-loop bandwidth should be a little higher than this—we'll see in Figure 8, which shows the closed-loop Bode magnitude plot.

The resonant peak of Figure 8 (about 3 dB at 0.11 Hz) is consistent with $\zeta = 0.5$, and the bandwidth (frequency where the closed-loop magnitude is 0 dB) is something above 0.1 Hz.

While this bandwidth may seem low, it's consistent with the desired z-plane pole locations of $0.45 \pm j0.45$ (where $f_n = 0.1013$ Hz), and the sampling frequency of 1 Hz.

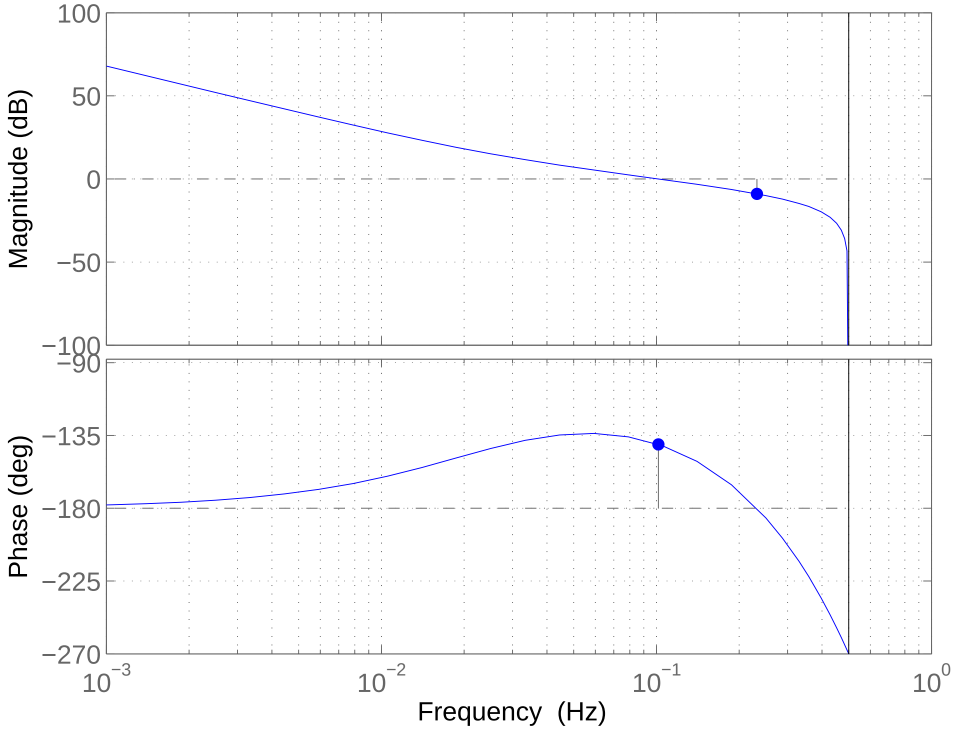


Figure 7: Bode plots of open-loop transfer function $KD(z)G(z)$.

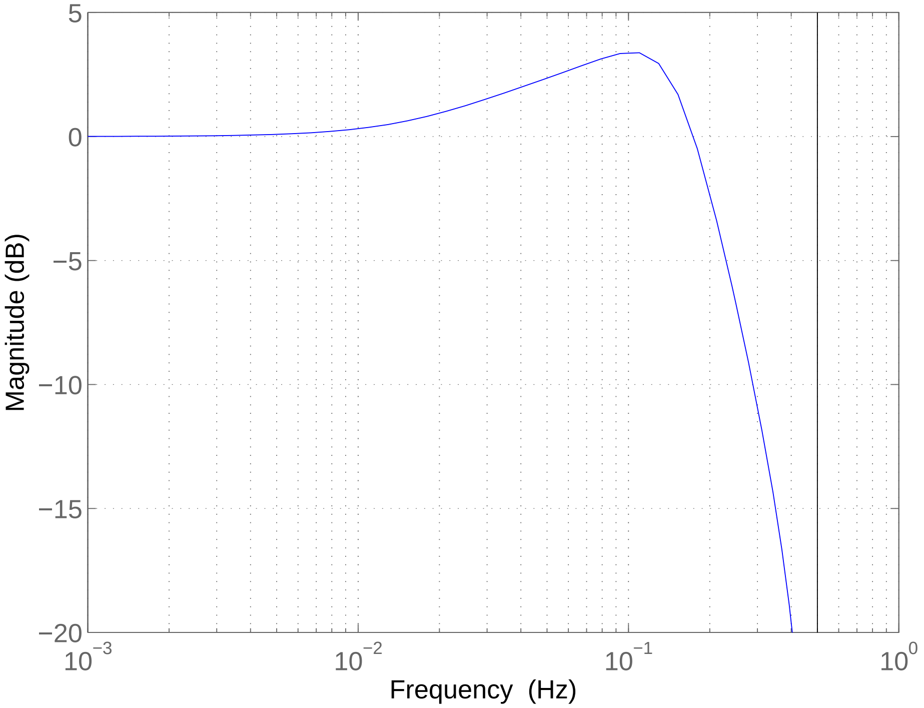


Figure 8: Bode plots of closed-loop transfer function.