Chapter 2 HW Solution

Problem 1. We are given the difference equation $y_k = 0.5y_{k-1} + 0.5y_{k-2} + 0.25u_{k-1}$.

(a) The discrete transfer for this system may be found by $\mathcal{Z}$-transforming the difference equation:

$$Y(z) = 0.5z^{-1}Y(z) + 0.5z^{-2}Y(z) + 0.25z^{-1}U(z) \implies \frac{Y(z)}{U(z)} = \frac{0.25z^{-1}}{1 - 0.5z^{-1} - 0.5z^{-2}} = \frac{0.25z}{z^2 - 0.5z - 0.5} \tag{1}$$

(b) One possible block diagram for this system is shown in Figure 1 below. There are other possibilities.

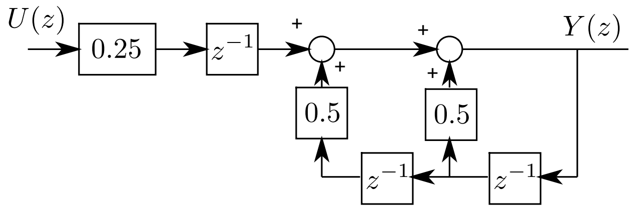


Figure 1: Block diagram of (1).

(c) The block diagram of Figure 1 *does* reduce to the yield the transfer function of (1); I'll show this in class if possible.

Problem 2. In this problem we're taking the "trapezoidal integration" example in my notes and extending this concept to a somewhat more rigorous integration rule. Instead of fitting a straight line between two samples and integrating the resulting trapezoid, we're fitting a parabola between *three* samples and integrating under that parabola over the last two points.

As I indicated in my "hints," the problem is algebraically much easier if you use the three points centered around $t = 0$: samples at $-T$, 0, and T seconds (time $t = 0$ is arbitrary).

(a) Refer to Figure 2 at right. As in the trapezoidal case, use u_{k-1} to be the integral of $e(t)$ up to time $t - T = -T$ (sample $k - 1$), hence as before we have

$$u_k = u_{k-1} + A \tag{2}$$

where A is the area between 0 and T shown in Figure 2. Using the three samples shown in Figure 2, along with fitted parabola $\hat{e}(t)$:

$$\hat{e}(t) = a_0 + a_1 t + a_2 t^2, \tag{3}$$

we fit the parabola by requiring

$$\hat{e}(-T) = a_0 - a_1 T + a_2 T^2 = e_{-1} \tag{4}$$

$$\hat{e}(0) = a_0 = e_0 \tag{5}$$

$$\hat{e}(T) = a_0 + a_1 T + a_2 T^2 = e_1. \tag{6}$$

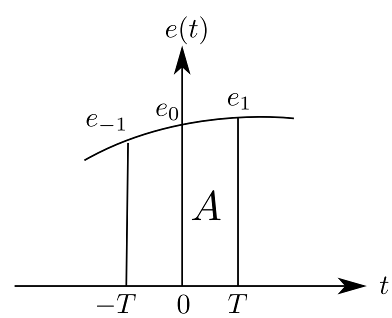


Figure 2: Parabolic integration.

Solving (4)-(6) simultaneously (not too hard) we get

$$a_0 = e_0, \qquad a_1 = \frac{e_1 - e_{-1}}{2T}, \qquad a_2 = \frac{e_{-1} - 2e_0 + e_1}{2T^2} \tag{7}$$

Next integrate parabola (3) to find area A (easier using the form of (3)):

$$A = \int_0^T \left(a_0 + a_1 t + a_2 t^2\right) dt = a_0 T + \frac{a_1 T^2}{2} + \frac{a_2 T^3}{3} \tag{8}$$

Substitute the expressions from (7) into (8), and after some manipulation you get area A as

$$A = \left[\frac{5e_1 + 8e_0 - e_{-1}}{12}\right] T \tag{9}$$

Now we generalize from samples -1, 0, 1 to samples $k-2$, $k-1$, and k, respectively. Doing this with equation (9) and substituting in equation (2) we get the final result for the difference equation for parabolic integration, which is...

$$\boxed{u_k = u_{k-1} + \frac{T}{12}\left[5e_k + 8e_{k-1} - e_{k-2}\right]} \tag{10}$$

(b) Applying the $\mathcal{Z}$-transform to (10) yields

$$U(z) = z^{-1}U(z) + \frac{T}{12}\left[5E(z) + 8z^{-1}E(z) - z^{-2}E(z)\right] \tag{11}$$

from which we find the discrete transfer function to be

$$\boxed{\frac{U(z)}{E(z)} = \frac{5T}{12}\left[\frac{z^2 + 1.6z - 0.2}{z(z-1)}\right]} \tag{12}$$

This transfer function has zeros at $z = 0.1165, -1.7165$, and poles at $z = 0, 1$.

(c) We are given sinusoid $e(t) = \sin 2\pi t$, which has frequency $f = 1$ Hz ($\omega = \pi$ rad/s). The first half-cycle is from 0 to 0.5 seconds, and with time step $T = 0.1$ second we have six steps (counting zero). The parabolic rule is given by (10), while recall the trapezoidal rule is

$$u_k = u_{k-1} + \frac{T}{2}\left[e_k + e_{k-1}\right] \tag{13}$$

Both of the difference equations were simple enough (and there were few enough samples) that I just used my calculator to work out the results, which are shown in Table 1 below (all samples for $k < 0$ assumed zero):

k	t	e_k	u_k
0	0.0	0	0
1	0.1	0.5878	0.0294
2	0.2	0.9511	0.1063
3	0.3	0.9511	0.2014
4	0.4	0.5875	0.2783
5	0.5	0	0.3077

k	t	e_k	u_k
0	0.0	0	0
1	0.1	0.5878	0.0245
2	0.2	0.9511	0.1033
3	0.3	0.9511	0.2014
4	0.4	0.5875	0.2814
5	0.5	0	0.3126

Table 1: Trapezoidal (left) and parabolic (right) integrators.

The exact result is:

$$\int_0^{0.5} \sin 2\pi t dt = \left[\frac{-1}{2\pi}\cos 2\pi t\right]_0^{0.5} = \frac{1}{\pi} = 0.3183 \tag{14}$$

Both integration rules are a little low, but the parabolic rule performs better (as one would expect). There are many far more accurate integration rules, but even the parabolic method is "overkill" for most real-time application.

Problem 3. We are given a "standard" 2^{nd} order underdamped transfer function:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{15}$$

The poles of this transfer function are at $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$.

We know that s-plane poles map to the z-plane using $z = e^{sT}$. When this mapping is applied to the poles of (15) one can express the resulting z-plane poles as $z = a \pm jb$. We would like to relate ζ and ω to the z-plane location a and b, and sampling period T.

As in the hints, it is much easier to work with the magnitude and angle of the z-plane pole locations (*i.e.* the polar form), given by

$$r = \sqrt{a^2 + b^2} \tag{16}$$

$$\theta = \tan^{-1}\frac{b}{a} \quad \text{(you would actually use atan2}(b,a)\text{ here)} \tag{17}$$

The mapping will yield

$$z = e^{sT} = e^{-\zeta\omega_n T} e^{\pm j\omega_n\sqrt{1-\zeta^2}T} = re^{j\theta} \tag{18}$$

The magnitude and angle of (18) are given by (take the "+" sign of the imaginary part)

$$r = e^{-\zeta\omega_n T} \tag{19}$$

$$\theta = \omega_n\sqrt{1-\zeta^2}T \tag{20}$$

Taking the natural log of (19) and squaring yields

$$(\zeta\omega_n T)^2 = \ln^2 r \tag{21}$$

Likewise, squaring and rearranging (20) yields

$$(\omega_n T)^2 - (\zeta\omega_n T)^2 = \theta^2 \tag{22}$$

Finally, solve (21) and (22) for ζ and ω_n and you'll get

$$\zeta = \frac{\ln r}{\sqrt{\ln^2 r + \theta^2}} \tag{23}$$

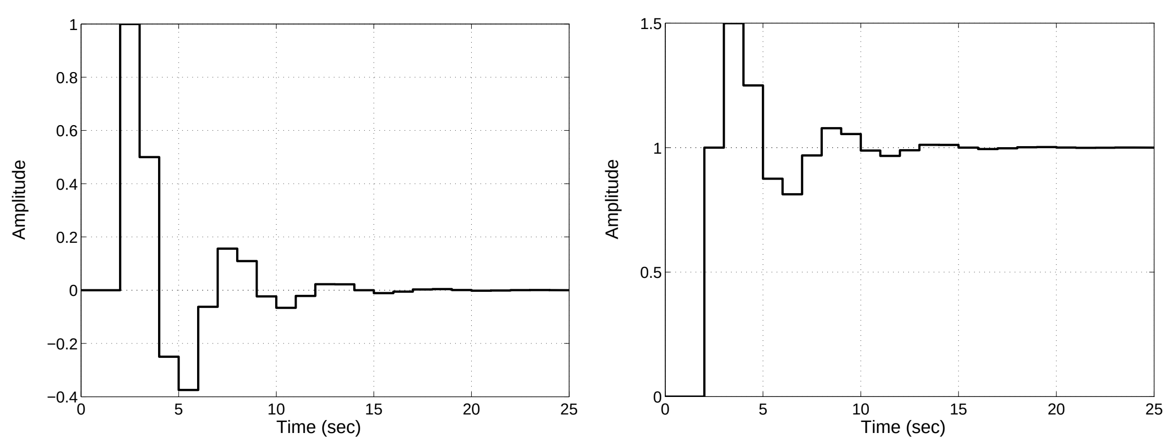
$$\omega_n = \frac{\sqrt{\ln^2 r + \theta^2}}{T} \tag{24}$$

where r and θ are related to a and b by (16) and (17).

Problem 4. We are given discrete transfer function $G(z) = \dfrac{1}{z^2 - 0.5z + 0.5}$.

(a) and (b) The unit pulse response can be found using MATLAB `impulse(sys)` and the unit step response using `step(sys)`. Discrete LTI system "`sys`" can be formed using the MATLAB `tf(num,den,T)` function or by defining `z = tf('z',T)` and typing in $G(z)$ directly. Since `T` must be specified, I used $T = 1$ sec.

The two plots are shown below. From Figure 3(b) it's clear that the DC gain is 1.



(a) Unit pulse response. (b) Unit step response.

Figure 3: Polynomial responses of discrete system.

(c) Now consider a sinusoidal input of frequency $f = 0.5$ Hz. The amplitude of the sinusoid is not given, so we'll assume unity. This input is given by

$$u(t) = \sin(2\pi f t) = \sin \pi t \tag{25}$$

Since we want to find the steady-state behavior of this system, I'll use three periods of this sinusoid as the input; the period is $\tau = 1/f = 2$ sec. Thus the total time of the simulation will be 6 seconds. With $T = 0.2$ (5 Hz) this corresponds to 30 samples. Below is the MATLAB code to produce the input and the response:

```
>> T = 0.2;         % Define the sampling period
>> w = pi;          % Set the input sinusoid frequency in rad/s
>> t = 0:T:6;       % Create the vector of sample times (from 0 -> 6 seconds)
>> u = sin(w*t);    % Create the input, could also use u = sin(w*k*T)
>> num = [0 0 1];        % Specify the numerator of the discrete transfer function
>> den = [1 -0.5 0.5];  % Specify the denominator
>> G = tf(num,den,T);   % Create the discrete LTI system
>> y = lsim(G,u,t);     % Get output, could also use lsim(G,u,t) which will plot
>> plot(t,u,'k-o',t,y,'r--*');  % Plot the input and output (not needed if lsim(G,u,t) used)
>> legend('Input','Output');    % Put a legend on the plot
```

The plot of the input and output is shown in Figure 4 below.

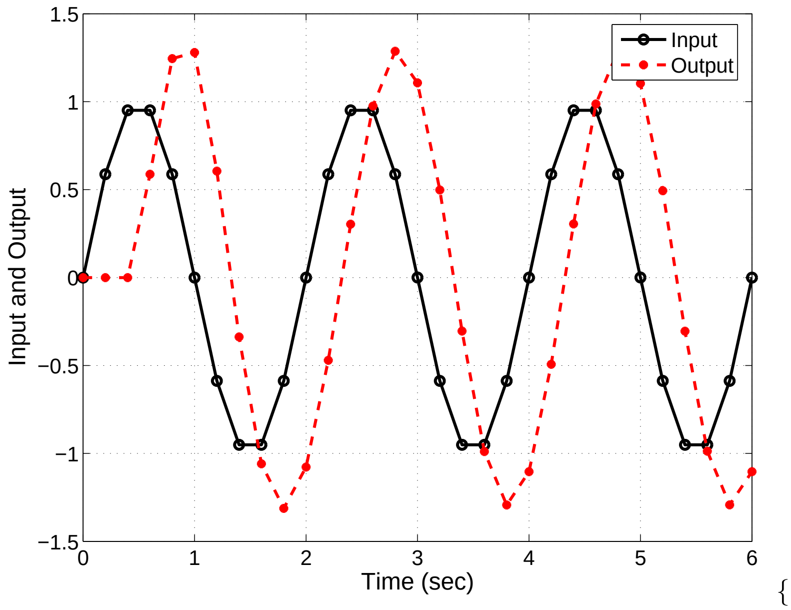


Figure 4: Sinusoidal input and output.

From Figure 4 we need to identify the amplitude ratio and phase shift. The amplitude of the input is of course unity, and I estimated the magnitude of the output to be about 1.3; the amplitude ratio (magnitude) is about 1.3. The *time shift* on the plot is about $t_\phi \approx -0.3$ seconds, and phase angle is related to time by $\phi = \omega t_\phi$, so the *phase shift* is $\phi \approx -0.3\pi$, or about -0.94 rad.

The amplitude ratio and phase can also be obtained by evaluating at frequency ω by finding $|G(j\omega T)|$ and $\angle G(j\omega T)$, respectively. Using MATLAB, I found

$$z = e^{j\omega T} = 0.8090 + j0.5878 \tag{26}$$

and substituting this z into $G(z)$ and evaluating the magnitude (`abs`) and angle (`angle`):

$$|G(z)| = |0.6793 - j1.1036| = 1.2959 \tag{27}$$
$$\angle G(z) = -1.019 \text{ rad} \tag{28}$$

Pretty close! FYI, the MATLAB commands to evaluate this single frequency response are shown on the next page.

Since we already have discrete LTI transfer function $G(z)$ in MATLAB,

```
>> G

Transfer function:
        1
-----------------
z^2 - 0.5 z + 0.5

Sampling time: 0.2
```

we can easily evaluate the magnitude and angle of $G(z)$ at a given frequency using the MATLAB `freqresp` function. Here's how:

```
>> f = 0.5;       % Frequency at which to get frequency response
>> w = 2*pi*f;    % Convert to rad/sec
>> Gfreqresp = freqresp(G,w)  % This will compute magnitude and phase at a given frequency

Gfreqresp = 0.6793 - 1.1036i  % This 'rectangular form' complex number represents the response
```

We can then find the magnitude and angle of this complex number:

```
>> mag = abs(Gfreqresp)  % The abs function computes the magnitude

mag = 1.2959

>> phase = angle(Gfreqresp)  % The angle function computes the phase (angle)

phase = -1.0190
```

It's rare to use MATLAB to get the frequency response at a single frequency (usually we'll use `bode` to get a range of frequencies) but it can be done.