Z-Plane RL Design using MATLAB

1 Introduction.

This document is intended to give you an example of using MATLAB and root locus to design controllers (compensators) in the z-plane.

2 Plant and Specifications.

2.1 Plant.

The plant will be an pure unit inertia; this could be any rotary element in a low-friction environment. The general transfer function from applied torque τ to angular position θ for a pure inertia is

$$\frac{\theta(s)}{\tau(s)} = \frac{1}{Js^2} \implies \frac{1}{s^2} \text{ (unit inertia)} \tag{1}$$

2.2 Specification.

I arbitrarily selected the closed-loop natural frequency to be 100 Hz and damping of 0.707, along with a sampling frequency of f_s=500 Hz (T=0.002). This corresponds to desired s-plane and z-plane pole locations of

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \approx -444 \pm j444. \tag{2}$$

$$z = e^{sT} = 0.2596 + j0.3192 \approx 0.26 + j0.32 \tag{3}$$

This sampling frequency is quite low; we would typically sample at perhaps 20× the bandwidth (natural frequency).

2.3 Discretized Plant.

With T=0.002, the zoh-discretized model of the plant (found using MATLAB c2d) is:

```
>> Gc = tf([0 0 1],[1 0 0])

Transfer function:
 1
---
s^2

>> Gd = c2d(Gc,T)

Transfer function:
2e-06 z + 2e-06
---------------
 z^2 - 2 z + 1

Sampling time: 0.002
```

This can be expressed as

$$G(z) = \frac{(2e-6)z + (2e-6)}{z^2 - 2z + 1} = \frac{(2e-6)(z+1)}{(z-1)^2} \tag{4}$$

Note the small numerator constant of $2e-6$; that will probably be "counteracted" by a large forward path gain.

3 Compensator Design.

Proportional control will be tried first, then a lead compensator.

3.1 Proportional Control.

With proportional control the actuating signal sent to the plant is *proportional* to the error signal, hence the "compensator" is a pure gain K.

3.1.1 Closed-Loop Transfer Function.

With system input called $R(z)$ and system controlled variable (output) called $Y(z)$ the closed-loop transfer function $W(z)$ is

$$\frac{Y(z)}{R(z)} = \frac{KG(z)}{1 + KG(z)} \tag{5}$$

hence the system characteristic equation is

$$1 + KG(z) = 0 \tag{6}$$

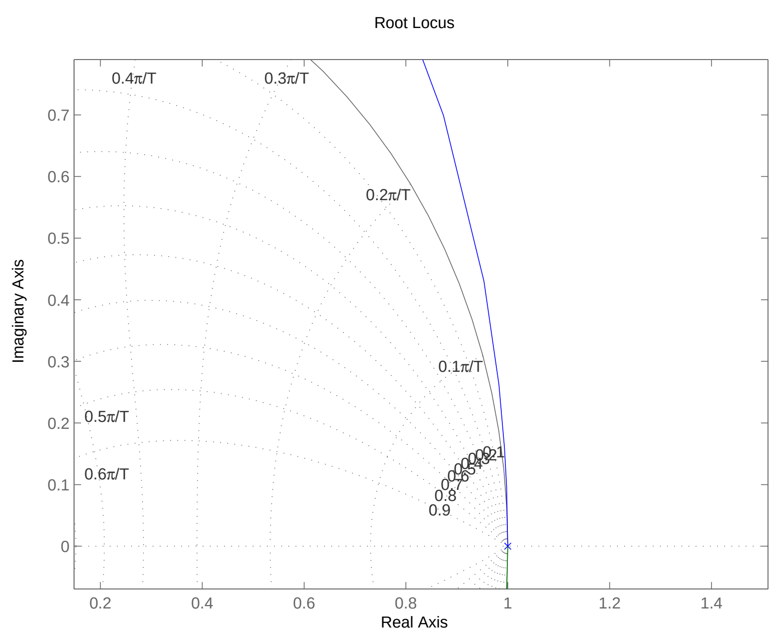
This is already in "root locus" form.

3.1.2 Root Locus of Proportional Control.

Using equation (6) we can draw the MATLAB root locus:

```
>> rlocus(Gd); axis equal; grid;
```

Zooming in on the locus near the "+1" point we get the figure below:



The locus (blue line) starts from the double pole at $z = 1$ and goes outside the unit circle—this system is unstable.

3.2 Lead Compensator.

We need to improve the stability of this system. A lead compensator (a quasi-differentiator) will do that. This can be seen from two perspectives:

- A lead network will pull the locus to the left, thus keeping it inside the unit circle
- A lead network adds positive phase angle, thereby improving the phase margin

The structure of a lead compensator $D(z)$ is

$$D(z) = \frac{z+b}{z+a} \tag{7}$$

where the zero is closer to the +1 point than the pole.

3.2.1 Compensator Pole/Zero Locations.

My philosophy on lead compensator design is to place the compensator zero in the s-plane at about one-third the distance of the desired dominant poles. Thus if the desired poles are at $-445 \pm j445$ I'd place the compensator zero at about $-445/3 \approx -150$. Now in the z-plane this zero location would be

$$z = e^{sT} = e^{(-150)(0.002)} \approx 0.74 \implies D(z) = \frac{z-0.74}{z+a} \tag{8}$$

The location of the compensator pole is found using the root-locus *angle condition*, in which we use open-loop transfer function $D(z)G(z)$:

$$\Sigma(\text{angles of vectors from poles to desired pole}) - \Sigma(\text{angles of vectors from zeros to desired pole}) = \pm 180^\circ \tag{9}$$

The relevant values in this problem are:

- The open-loop poles: -1, -1, and the unknown lead compensator pole
- The open-loop zeros: 0.74, -1
- The desired closed-loop pole: 0.26+j0.32

I suggest you draw the z-plane and draw these vectors. Here are the values that I found:

$$2(156^\circ) + \theta - 146^\circ - 14^\circ = \pm 180^\circ \tag{10}$$

Parameter θ in (10) is the angle of the vector from the lead compensator pole to the desired closed-loop pole location. From (10) this angle is

$$\theta = 28^\circ \tag{11}$$

This implies that the pole must be located at $z = -0.354$, thus the lead compensator is

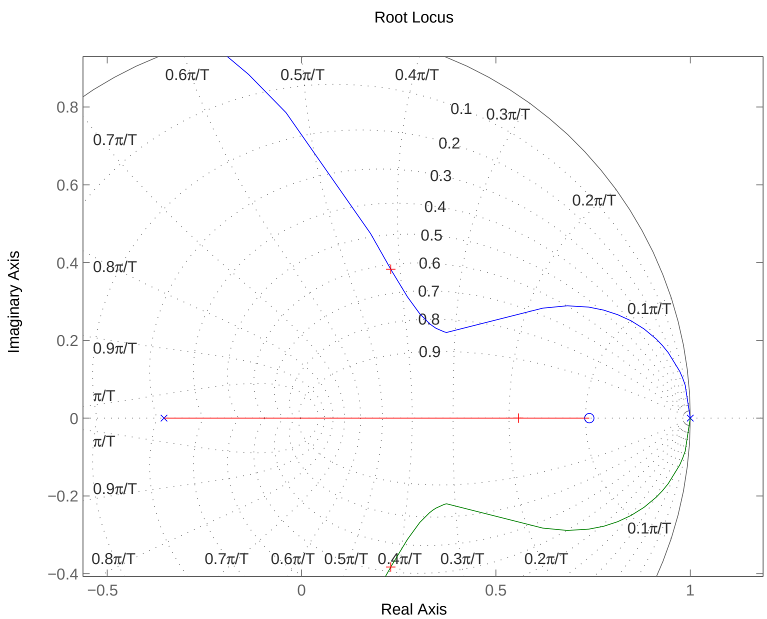
$$D(z) = \frac{z-0.74}{z+0.35} \tag{12}$$

The root-locus diagram of $D(z)G(z)$ is shown on the next page; the "cross" shows where I specified the desired pole (near $\zeta = 0.6$).

The corresponding K at this point is:

$$K = 3.14e5 = 314,000 \tag{13}$$

Remember the small numerator constant in $G(z)$? This K makes up for it.



4 Final Closed-Loop System.

With the $D(z)$ and $G(z)$ given previously, the closed-loop transfer function is

$$\frac{Y(z)}{R(z)} = \frac{KD(z)G(z)}{1 + KD(z)G(z)} \tag{14}$$

Using MATLAB, this is

```
>> Wd = feedback(K*Dd*Gd,1)

Transfer function:
  0.6287 z^2 + 0.1635 z - 0.4652
----------------------------------
z^3 - 1.017 z^2 + 0.4555 z - 0.1112

Sampling time: 0.002
>> damp(Wd)

       Eigenvalue          Magnitude    Equiv. Damping    Equiv. Freq. (rad/s)

  5.58e-01                  5.58e-01       1.00e+00             2.91e+02
  2.29e-01 + 3.83e-01i      4.46e-01       6.16e-01             6.54e+02
  2.29e-01 - 3.83e-01i      4.46e-01       6.16e-01             6.54e+02
```

The complex closed-loop poles have a damping ratio of 0.65 and a natural frequency of 654 rad/s (104 Hz). I'd say our design is pretty close.

One could get the step response, *etc.*

Since this is a Type II system, it should definitely have a DC gain of 1.00; from MATLAB we get

```
>> dcgain(Wd)

ans = 1
```