A Simple Essay on Complex Numbers

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1 Introduction

Several classes with which I'm involved require a working knowledge of complex numbers. I find that many students are somewhat "uneasy" with complex numbers. A typical classroom interaction is something like this:

> "So, how many of you folks are completely at home with positive numbers?" (most all the class raises their hands)...
>
> "Good...now how many of you are completely at home with negative numbers?" (sensing a "trick question," fewer hands go up this time, but still quite a few)...
>
> "Okay...now how many of you are completely at home with *imaginary* numbers?" (this time nobody raises their hand...occasionally one brave soul, but usually not)...

And therein lies the problem. Complex numbers are *very useful*, but most students are ignorant of their true nature and hence wary of them.

The purpose of this little essay is to present a gentle and non-threatening introduction to complex numbers. So let's get started...

2 Positive Numbers

You are all undoubtedly confident with the concept of positive numbers. You can add them, subtract them, multiply, and so on. Positive numbers can be displayed on the *number line*, as shown in Figure 1.

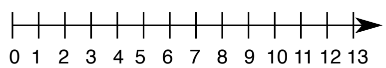

Figure 1: Positive number line.

You can easily plot a positive number on the number line. No problem.

3 Negative Numbers

Now things get a little trickier. Hmm, negative numbers. How do you explain a concept like negative numbers? Let's say you have the negative number -2 and you're trying to explain it to one of the residents of the planet Ogg. You say to the Oggian: "well, it's like if you have -2 scruzzes, and you add 2 scruzzes, you end up with nothing." The Oggian scratches his head (well, assuming they *have* heads), and is still confused.

Let's try it another way. *I* claim that the *only* reason to have a bizarre concept like negative numbers is so you can write down the answer to equations like:

$$x + 2 = 0. \tag{1}$$

If you don't have the concept of negative numbers you can't solve this equation. But—evidently unlike the poor residents of Ogg— we *do have* the concept of negative numbers, so we confidently express the solution as

$$x = -2 \tag{2}$$

3.1 The "-" Operator

In equation (2) we must pay *particular* attention to this "-" thing, which is actually an *operator*—it *operates* on 2 to transform it to -2. And we *know* that the "-" is an operator—you would never think that the "-" by itself has a numerical value, it just negates numbers; it's the negation operator.

Negative numbers extend out to the left on the *number line*, shown in Figure 2.

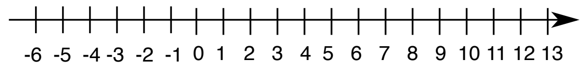

Figure 2: Positive and negative number line.

You can plot a negative number on the number line of Figure 2. But consider *this* interpretation of what the "-" operator does: In the process of negating a number, the "-" operator actually *rotates* the number by 180° (ccw) around the 0 point. Try it! Put your compass point on 0, and the pencil part on a number, then give it a 180° ccw rotation...you've just negated that quantity! And if you negate an already-negative number, it ends up being positive! Just like $-(-2) = 2$. This "rotational" interpretation is critical for your understanding of the *imaginary* operator, which is our next step.

4 Imaginary Numbers

Okay...if you really understand the "-" operator and negative numbers, then I claim that imaginary numbers are actually easier...

Just like negative numbers, imaginary (terrible name, "imaginary" numbers are no less real or significant than "real" numbers) numbers must exist so we can solve equations like:

$$x^2 + 4 = 0 \tag{3}$$

Without the concept of imaginary numbers you cannot write down the solution to (3). And *that* is the only reason for the existence of imaginary numbers. *With* the concept of imaginary numbers, the solution (ignoring ± sign) is:

$$x = j2 \tag{4}$$

Note that one should place the "j" operator *before* the number, just like the "-" operator. This affirms the "j" is an *operator* and not a *number*. Unfortunately many times this is not done. Finally, mathematicians use i, but we use j because i is electrical current.

4.1 The "j" Imaginary Operator

Just like the negation operator "-", the imaginary operator "j" takes a number and "imaginates" it. And imaginary numbers are plotted along an axis that is perpendicular (orthogonal is the classy word) to the real axis of Figures 1 and 2. This is shown in Figure 3 in what has now become the *number plane*.

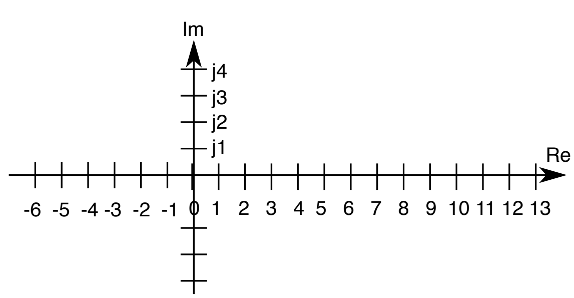

Figure 3: The number plane.

We plot positive real numbers to the right, negative real numbers (180° rotation) to the left, and positive imaginary numbers *up*, and negative imaginary numbers (not annotated) *down*.

Note that the imaginary operator "j" performs a 90° ccw rotation in the number plane—it's only half as difficult as negation! And if you operate *twice* by the imaginary operator you get the "-" operator—some people then make the assignment

$$j \cdot j = -1 \implies j = \sqrt{-1}$$

I think this is **highly misleading**: you wouldn't consider the negation operator "-" to have a value; in the same way you shouldn't think of the imaginary operator "j" to have a value. So please dont think of $j = \sqrt{-1}$.

5 Complex numbers

The most general kind of number is a *complex* number, which has a real part (`Re`) and an imaginary part (`Im`). Such a number is written

$$2 + j3 \tag{5}$$

and this number can be located on the number plane of Figure 3.

We also have complex *variables*, which also have `Re` and `Im` parts, like

$$s = \sigma + j\omega \tag{6}$$

in which σ is the `Re` parts, and ω (without the "j") is the `Im` part.

5.1 Rectangular and polar forms.

The number of (5) and variable of (6) are in *rectangular* form, in which the `Re` and `Im` parts are explicit. The other common form is the *polar* form, which consists of a *magnitude* and an *angle*. Complex numbers and variables can be expressed in polar form in two ways; consider a number with magnitude of 3 and angle of 45°:

$$3\angle 45° \text{ or } 3e^{j(\pi/4)} \tag{7}$$

The first form of (7) is a "shorthand" while the second "exponential" form is mathematically rigorous.

The "exponential" form comes from Euler's famous identity, which is

$$\boxed{e^{j\theta} = cos\theta + j\sin\theta} \tag{8}$$

This unlikely relationship can be verified using the series expansion of all terms (try it!).

Given complex number $a + jb$ we can use (8) to find the corresponding polar form $re^{j\theta}$:

$$r = \sqrt{(a^2 + b^2)}, \;\; \theta = \tan^{-1}\frac{b}{a} \tag{9}$$

NOTE: One should always use a "four-quadrant" arctangent function (usually called `atan2(Im,Re)`) to guarantee finding the correct angle θ.

Given complex number $re^{j\theta}$, the rectangular form $a + jb$ is given by

$$a = r\cos\theta, \;\; b = r\sin\theta \tag{10}$$

5.2 Complex number arithmetic

The rectangular form is useful for addition (and subtraction), while the polar form is more convenient for multiplication and division.

5.2.1 Addition and subtraction

Given $s_1 = a_1 + jb_1$ and $s_2 = a_2 + jb_2$, the sum is given by

$$s_1 + s_2 = (a_1 + a_2) + j\,(b_1 + b_2) \tag{11}$$

Subtraction is similar...

5.2.2 Multiplication and division

Here the polar form is more convenient. Consider $s_1 = r_1 e^{j\theta_1}$ and $s_2 = r_2 e^{j\theta_2}$. The product is given by

$$s_1 s_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)} \tag{12}$$

The magnitudes multiply while the angles add.
Similarly, division is done by

$$\frac{s_1}{s_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} \tag{13}$$

6 Conclusion.

Well, that's about it. Perhaps this can help reduce your anxiety regarding complex numbers...
Please feel free to critique this document to me—I may use it for future classes of mine (not just digital control).