Chapter 8 HW Solution

Review Questions.
1. What is a root locus? A plot of the possible closed-loop pole locations as some parameter varies from 0 to \( \infty \).
4. Do the zeros of a system change with a change in gain? No.
5. Where are the zeros of the closed-loop transfer function? They are the roots of the numerator of the closed-loop transfer function.
7. How can you tell from the root locus if a system is unstable? If any of the poles are in the right-half \( s \)-plane.

Problem 1. The potential root loci for the stated parts are shown below:

![Potential root loci for Problem 1](image)

(a) CANNOT be root locus: not symmetric about Real axis, and real axis branches not shown.
(b) CANNOT be root locus: Real axis branch is to the left of FOUR poles/zeros (even number).
(e) CANNOT be root locus: not symmetric about Real axis, and no real axis branch between zeros.
(g) CANNOT be root locus: not symmetric about Real axis (poles must occur in conjugate pairs), and real axis branch is to the left of TWO poles/zeros (even number).
(h) CAN be root locus: everything looks correct to me.

Problem 2. Do parts (a), (b), and (c). In accordance with my comments, my drawing attempts are shown below:
Problem 8. The characteristic equation (denominator of the closed-loop transfer function set equal to zero) is:

\[ s^3 + 2s^2 + (20K + 7)s + 100K = 0 \]  
\[ (1) \]

When this CE is put into “root locus” for versus parameter \( K \), it is

\[ 1 + K \frac{20(s + 5)}{s(s^2 + 2s + 7)} = 0 \]
\[ (2) \]

In the process of sketching the root locus, I asked for the following:

(a) **Real-axis branches.** As shown on the sketch, there is a real axis branch between the pole at \( s = 0 \) and the zero at \( s = -5 \).

(b) **Asymptote intersection and angle.** Since there are 3 poles and 1 zero, there are 2 asymptotes. The asymptote angle and intersection are:

\[ \theta_a = \pm \frac{180^\circ}{3} = \pm 90^\circ, \quad \sigma_a = \frac{3 - 1 - (-5)}{3 - 1} = \frac{3}{2} = 1.5 \]
\[ (3) \]

(c) **Departure angle from complex poles.** Let \( s \approx -1 + j2.45 \) (near the upper complex pole), and evaluate the angle condition (angle of vectors from poles to \( s \) minus angle of vectors from zeros to \( s \)):

\[ 112.2^\circ + 90^\circ - 31.5^\circ = \pm 180^\circ \implies \theta = 9.3^\circ \]
\[ (4) \]

(d) **Imaginary axis crossing point and corresponding \( K \).** Substituting \( s = j\omega \) into the characteristic equation of (1), you can find

\[ j\omega(-\omega^2 + j2\omega + 7) + K[20(j\omega + 5)] = 0. \]
\[ (5) \]

so

\[ j\omega^3 - 2\omega^2 + j7\omega + j20K\omega + 100K = 0. \]
\[ (6) \]

Equation (6) yields a Real equation and an Imag equation:

- Real: \( -2\omega^2 + 100K = 0 \)
  \[ (7) \]
- Imag: \( -\omega^3 + 7\omega + 20K\omega = 0 \)
  \[ (8) \]

Solving (7) and (8) simultaneously (not hard), you find that

\[ K = \frac{7}{30} = 0.2333, \quad \omega = \sqrt{50K} = 3.42 \]
\[ (9) \]

My root locus sketch is shown in Figure 2 on the next page.
Problem 30. A unity-feedback system is shown in the block diagram below:

a. Find $\alpha$ so the system will have a settling time of 4 seconds for large values of $K$. The expression for 2% settling time $T_s$ is

$$T_s = \frac{4}{\zeta \omega_n} = 4 \text{ sec} \implies \zeta \omega_n = 1,$$

where $\zeta \omega_n$ is the REAL part of a complex pole location. The closed-loop TF is

$$\frac{C(s)}{R(s)} = \frac{K(s + \alpha)}{s(s + 1)(s + 10)} + \frac{K(s + \alpha)}{s(s + 1)(s + 10)} = 0$$  \hspace{1cm} (11)

The corresponding characteristic equation and root locus form are

$$s(s + 1)(s + 10) + K(s + \alpha) = 0 \implies 1 + K\frac{s + \alpha}{s(s + 1)(s + 10)} = 0$$  \hspace{1cm} (12)

The root locus diagram will have 3 poles and 1 zeros, hence TWO asymptotes at angles of $\pm 90^\circ$. The complex RL branches will converge to these asymptotes, so if these asymptotes intersect the REAL axis at $s = -1$, the settling time condition of (10) will be satisfied.

The equation for asymptote intersection is

$$\sigma_a = \frac{\Sigma \text{pole locations} - \Sigma \text{zero locations}}{\# \text{poles} - \# \text{zeros}} = \frac{0 - 1 - 10 - (-\alpha)}{3 - 1} = \frac{-11 + \alpha}{2} = -1.$$  \hspace{1cm} (13)

Solving (13) for $\alpha$ yields

$$\alpha = 9$$ \hspace{1cm} (14)
b. Find gain \( K \) such that the complex closed-loop poles have damping ratio \( \zeta \approx 0.5 \). With \( \alpha = 9 \), the RL form of the characteristic equation is

\[
1 + K \frac{s + 9}{s(s + 1)(s + 10)} = 0
\]  

(15)

A portion of the \textit{MATLAB} root locus is shown below, with the pole locations indicated when \( K \approx 1.14 \) (this is approximately where \( \zeta = 0.5 \)).

Poles at \( s = -0.5 \pm j0.88, -9.98 \) when \( K = 1.14 \)

\[ \begin{align*}
C(s) &= \frac{1.14(s + 9)}{(s + 0.5 \pm j0.88)(s + 9.98)} \\
R(s) &= \frac{1.14(s + 9)}{(s^2 + s + 1.02s)(s + 9.98)}
\end{align*} \]

(16)

where the second TF in (16) has the complex terms multiplied together to form a quadratic.

The unit step response is shown in Figure 3 on the next page.

d. Is the overshoot what you expect? Is the closed-loop system dominated by a pair of complex poles?

Yes, a damping ratio of \( \zeta = 0.5 \) should produce an overshoot of about 15%. And yes, the system response is dominated by a pair of complex poles. The complex poles of the root locus diagram are \textit{MUCH} closer to the origin than either the zero at -9 or the third pole at -9.98.

e. Is the settling time what the text problem statement requested? Why not?

No, the settling time is longer than 4 seconds. That’s because we \textit{did not} use a “large value of \( K \).”
Figure 3: Unit step response for Problem 30 with $K$ set for $\zeta = 0.5$.

f. Use the root-locus diagram of (a) to select a $K$ that should satisfy the settling time requirement. By “clicking” on the complex branch high enough so the real part is nearly -1, I found the following value for $K$ and the corresponding pole locations:

$$K \approx 199$$

poles at $s = -0.86 \pm j13.9, -9.2787$ \hspace{1cm} (17) \hspace{1cm} (18)


Figure 4: Unit step response for Problem 30 with “large” $K$ set for $T_s = 4$ Not too desirable.


g. Use either MATLAB or Simulink to plot the step response with the $K$ from (f). Does this step response look desirable? The unit step response is shown below. With this large $K$ there is a LOT of oscillation—doesn’t look desirable to me.
Problem 60. I modified the block diagram of the system to be more descriptive, as shown below:

![Block Diagram](image)

Figure 5: F4E Pitch Stabilization Loop.

Here is how this pitch angle control system is supposed to work:

- Block “Aircraft Dynamics” $G_2(s)$ has input of elevator angle $\delta_{\text{com}}$, and output pitch rate $\omega$. This pitch rate is simply the angular velocity of the aircraft as it pitches up/down, and has units of rad/s.
- The “inner loop” with gain $K_2$ is intended to control this pitch rate $\omega$. Therefore its input should be commanded pitch rate (rad/s), which I have labeled as $\omega_c$.
- The “outer loop” with gain $K_1$ is intended to control the pitch angle, which I have relabeled as $\theta$.
- This structure of an “inner loop” controlling velocity, and an “outer loop” controlling position is very common in motion control. At least this part of the problem was reasonable.

Behavior of “aircraft dynamics.” Consider the sketch shown below, where the $Z$ axis defines positive angle.

![Sketch](image)

The aircraft elevator is shown with a positive displacement $\delta_{\text{com}}$ (positive around $Z$ axis by RH rule). By intuitive reasoning you can see that this elevator displacement will cause the aircraft to pitch down, as shown by the angular direction $\omega$.

By the sense of positive angle, this resulting $\omega$ is in the negative direction. This is the reason for the “minus” sign in the numerator of “aircraft dynamics” transfer function $G_2(s)$. 
a. Simulate the behavior of the aircraft dynamics. Apply a 1° positive elevator angle (i.e. downward elevator angular displacement as shown in the sketch). Use MATLAB to plot the pitch rate $\omega$ (rad/s) vs time for 1 second. What is the pitch rate (rad/s) after 1 second? Does this seem reasonable?

The transfer function of the aircraft dynamics from commanded elevator angle input $\delta_{\text{com}}$ (rad) to pitch rate output $\omega$ (rad/s) is

$$G_2(s) = \frac{\omega(s)}{\delta_{\text{com}}(s)} = \frac{-508(s + 1.6)}{(s + 14)(s - 1.8)(s + 4.9)} \text{ rad/s}$$  \hspace{1cm} (19)

An input of 1° is $1/57.3$ radians, and the resulting step response is shown below.

![Figure 6: Aircraft pitch rate in response to 1° elevator angle.](image)

At time of 1.0 second, the pitch rate $\omega = -0.8415$ rad/s ($-48^\circ/$sec). For a ONE DEGREE elevator angle, that seems awfully large to me, so I would say that it doesn’t look reasonable, but that’s a judgment call...

b. Inner loop CL TF. Using standard block diagram reduction, the closed-loop transfer function of the inner loop is

$$\frac{\omega(s)}{\omega_c(s)} = \frac{-508K_2(s + 1.6)}{(s + 4.9)(s + 14)(s - 1.8) - 508K_2(s + 1.6)} \text{ rad/s}$$  \hspace{1cm} (20)

c. Inner loop RL form. The root-locus form of the inner-loop characteristic equation of (1) based on parameter $K_2$ is

$$1 - 508K_2\frac{s + 1.6}{(s + 4.9)(s + 14)(s - 1.8)} = 0$$  \hspace{1cm} (21)

d. Selection of inner loop gain. Define $K \triangleq -508K_2$, the inner loop RL form is now

$$1 + K\frac{s + 1.6}{(s + 4.9)(s + 14)(s - 1.8)} = 0$$  \hspace{1cm} (22)

Use MATLAB to draw the root locus diagram of (3), and select $K_2$ (actually $K$) to place the complex poles to yield a damping ratio of $\zeta = 0.5$. I found that $-0.4 > K_2 > -0.5$. The MATLAB root locus is shown on the next page in Figure 8.
In Figure 8, to place the complex poles with \( \zeta = 0.5 \) requires (recall \( K = -508K_2 \))

\[
K = 239 \implies K_2 = -0.471
\] (23)

The reason for the negative \( K_2 \) is because of the “minus” sign in the numerator of \( G_2(s) \) (downward pitching for positive elevator angle).

e. Factored form of inner loop TF. With the value of \( K_2 \) from part (d), the factored form of inner loop transfer function \( \frac{\omega(s)}{\omega_c(s)} \) (using the pole location from the root locus) is:

\[
\frac{\omega(s)}{\omega_c(s)} = \frac{239(s + 1.6)}{(s + 1.006)(s + 8.05 \pm j13.89)} \text{ rad/s}
\] (24)

Note that the inner loop is a Type 0 system (no free integration in forward path), and so the DC gain of (24) is NOT one, but is in fact about 1.47.

f. Response of controlled inner loop. Use either MATLAB or Simulink to find the step response of the controlled inner loop to a commanded pitch rate of \( \omega_c = 1^\circ/\text{sec} \). Plot the resulting pitch rate \( \omega \) (degrees/sec) versus time. Does it look reasonable? What is the final value of \( \omega \) in degrees/sec? Does this agree with the DC gain of the inner loop? Remember that a step input of \( 1^\circ/\text{sec} \) is \( 1/57.3 \text{ rad/sec} \). The response of the controlled inner loop is shown in Figure 9.
Figure 8: Response of controlled inner loop to desired pitch rate of 1°/sec.

I would say that the response of Figure 9 does look reasonable. The initial response quite fast. There is a slow “exponential” final convergence—this is the mode associated with the closed-loop pole at $s = -1$ in the inner-loop transfer function of (24).

In response to a desired pitch rate of 1°/sec, the actual aircraft pitch rate of Figure 9 goes to a final value of 1.47°/sec. This does agree with the inner loop DC gain of 1.47.

g. Block diagram of outer loop. Using the factored form of the inner loop from part (e), draw a block diagram of the outer loop (similar to Figure 1 on the previous page). The outer loop forward path will have three blocks in series: (1) gain $K_1$, (2) factored form of inner loop TF from part (e), and (3) the integrator 1/s. What is the TYPE of the outer loop system?

The block diagram is shown below:

Since there is an integration in the forward path (the integration from $\omega$ to $\theta$), this system is TYPE 1.

h. Closed-loop TF of outer loop. From the block diagram of part (g), find the closed-loop transfer function $\frac{\theta(s)}{\theta_c(s)}$.

Using the standard unity-feedback closed-loop reduction, it is

$$\frac{\theta(s)}{\theta_c(s)} = \frac{239K_1(s + 1.6)}{s(s + 1)(s^2 + 16s + 257) + 239K_1(s + 1.6)} \tag{25}$$

Note that I approximated the denominator term $(s + 1.006)$ by $(s + 1)$. As I’ve said before, we usually don’t know these parameters that accurately.
i. **Outer loop CE and RL form.** Find the characteristic equation of the outer loop, and put it in root-locus form using parameter $K_1$.

The CE of the outer loop, with corresponding root locus form are:

<table>
<thead>
<tr>
<th>CE: $s(s + 1)(s^2 + 16s + 257)) + 239K_1(s + 1.6) = 0$</th>
<th>(26)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RL form: $1 + K_1 \frac{239(s + 1.6)}{s(s + 1)(s^2 + 16s + 257)} = 0$</td>
<td>(27)</td>
</tr>
</tbody>
</table>

j. **Selection of outer loop gain.** Use MATLAB to draw the root-locus diagram of part (i) versus gain $K_1$. Select $K_1$ so the complex poles have a damping ratio of $\zeta = 0.45$.

The MATLAB root locus diagram of (27) is shown below (lower complex branch not shown):

![Root Locus Diagram](image)

The desired CL poles are where the line at $\zeta = 0.45$ intersects the complex branches. The gain at this point is $K_1 = 2.646$.

The closed-loop pole locations (from the RL diagram above) corresponding to that $K_1$ are:

$$s = -1.9 \pm j1.02 \quad (2 \text{ poles})$$

$$= -6.6 \pm j13.2 \quad (2 \text{ poles})$$

k. **Factored form of outer loop.** With the value of $K_1$ you selected in part (j), what is the factored form of the outer-loop transfer function (the entire system)? This will be $\frac{\theta(s)}{\theta_c(s)}$.

From the $K_1$ and the closed-loop poles of the above RL diagram, this will be

$$\frac{\theta(s)}{\theta_c(s)} = \frac{632(s + 1.6)}{(s + 1.9 \pm j1.02)(s + 6.6 \pm j13.2)} \text{ rad}$$

(30)
1. **Response of controlled outer loop.** Using Simulink, construct a model of the entire system. You will have the two nested loops. Use the two numerical values you’ve found for $K_2$ and $K_1$, plus all other numerical values, of course.

Let the commanded pitch angle input be a step of $1^\circ$, and produce the following two plots:

- Plot the actual pitch angle $\theta$ (degrees) of the aircraft versus time
- Plot the elevator angle $\delta_{\text{com}}$ (degrees) versus time

Does the response of the aircraft look reasonable? Does the response of the elevator angle look reasonable (you may have to think about that a little)?

**Simulink Model:** My Simulink block diagram is shown below:

**Response Plots:** The plots of elevator angle and aircraft pitch angle in response to a commanded pitch angle of $1^\circ$ are shown below:

![Response Plots](image)

(a) Elevator angle response.

(b) Pitch angle response.

Figure 9: Responses of pitch angle control system to $1^\circ$ input.

**Comments on response:**

- **Aircraft response (pitch angle):** The overall pitch angle of the aircraft responds pretty quickly—it is mostly all the way there in less than 1.0 second. And since this is a **Type 1** system, the final value goes to $1^\circ$ as desired...no steady-state error. I would say the aircraft response **DOES LOOK REASONABLE**.

- **Elevator angle:** The initial response of the elevator angle is strongly **negative** (look closely); this is to get the aircraft pitching **positively**. The elevator angle then quickly goes **positive** to stop the aircraft before it pitches too far. This is the same thing you would do as a human pilot of your intuition and reflexes were good enough. So again, the elevator response **DOES LOOK REASONABLE**.

- Whew! I’m tired...