Chapter 7 HW Solution

Review Questions.

1. Name two sources of steady-state errors.

1. System configuration (TYPE too low)
2. Type of applied input (ramp, parabolic, *etc.*

3. Name the test inputs used to evaluate steady-state error.

1. Step (constant position)
2. Ramp (constant velocity)
3. Parabolic (constant acceleration)

4. How many integrations in the forward path are required for there to be zero steady-state error for each of the test inputs listed in Question 3?

1. Step: need **ONE** integration
2. Ramp: need **TWO** integrations
3. Parabola: need **THREE** integrations

5. Increasing system gain has what effect upon the steady-state error? It usually reduces the error.

9. Define *system type*. Number of integrations in the forward path of a closed-loop system.

Problem 2. The ramp response of a system is shown in Figure 1 at right.

(a) The steady-state error can be read right from the plot after the "startup" transient has died out; it is

$$\boxed{e_{ss} = 2 \text{ units}} \tag{1}$$

(b) Now the input is a unit ramp $r(t) = t$. The input of Figure 1 was a ramp of slope

$$r(t) = v_o t = \frac{5}{2}t = 2.5t \tag{2}$$

With a ramp input of slope 2.5 the steady-state error was 2 units. Thus—for a linear system—with a ramp input of slope 1, the steady-state error will be

$$\boxed{e_{ss} = 2\left(\frac{1}{2.5}\right) = 0.8 \text{ units}} \tag{3}$$

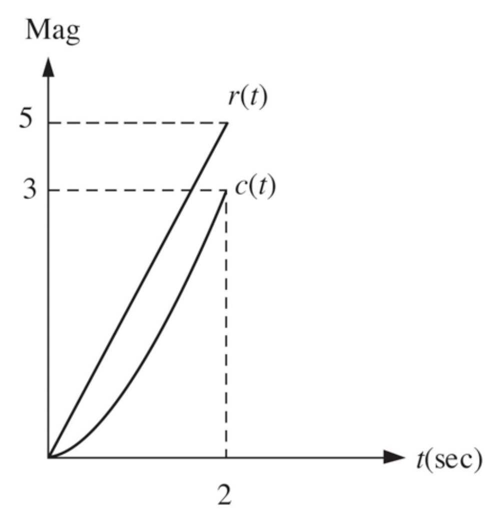

Figure 1: Ramp response.

Problem 10. The block diagram for this problem is shown below, where $G(s) = \dfrac{5000}{s(s+75)}$.

a. The closed-loop transfer funtion $T(s)$ for this system is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{5000}{s(s+75)+5000} = \frac{5000}{s^2+75s+5000} = \frac{K}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{4}$$

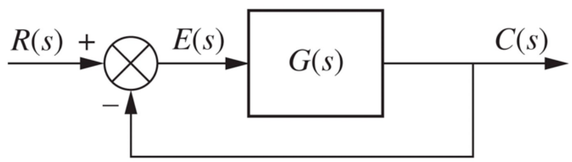

Figure 2: Unity feedback system of Problem 10.

It is easy to show that:

$$\zeta = 0.53 \quad (5)$$

$$\omega_n = 71 \text{ rad/s} \quad (6)$$

So since $\zeta = 0.53$, the percent overshoot to a step input is about

$$\boxed{\%\text{ OS} \approx 15\%} \quad (7)$$

b. The 2% settling time is about

$$\boxed{T_s \approx \frac{4}{\zeta\omega_n} = 0.11 \text{ sec}} \quad (8)$$

c. To find the steady-state error to a step input of magnitude 5, notice that the system is **TYPE 1**, therefore from Table 7.2 the step error constant $K_p = \infty$, and

$$\boxed{\text{For } r(t) = 5, \;\; e_{ss} = 0} \quad (9)$$

d. For a ramp input of $r(t) = 5t$, one needs the **velocity error constant** K_v, thus

$$K_v = \lim_{s=0} sG(s) = \lim_{s=0} \frac{(s)5000}{s(s+75)} = \frac{5000}{75} \quad (10)$$

Then the steady-state error to a ramp of slope $v_o = 5$ is

$$\boxed{e_{ss} = \frac{v_o}{K_v} = \frac{5}{5000/75} = 0.075} \quad (11)$$

e. For a parabolic input $r(t) = \frac{1}{2}a_o t^2 = 5t^2$, one uses the **acceleration error constant** K_a, and from Table 7.2 $K_a = 0$. Thus the error is

$$\boxed{e_{ss} = \frac{a_o}{K_a} = \frac{5(2)}{0} = \infty} \quad (12)$$

The steady-state error of ∞ means that the system keeps falling further and further behind.

Problem 15. The block diagram for this problem is shown below in Figure 3:

Since the system **TYPE** is the number of "free" integrations in the forward path, we need to reduce the system block diagram to the point where we have the forward path transfer function.

First reduce the inner loop,

$$G(s)_{\text{inner}} = \frac{100(s+2)}{s(s+5) + 1000(s+2)} = \frac{100(s+2)}{s^2 + 1005s + 2000)} \quad (13)$$

The complete forward path transfer function is the product of this $G(s)_{\text{inner}}$ with $\frac{1000}{s}$. Since $G(s)_{\text{inner}}$ has **ZERO** free integrations, and $\frac{1000}{s}$ has **ONE** free integration, the result is:

$$\boxed{\text{System TYPE} = 1} \quad (14)$$

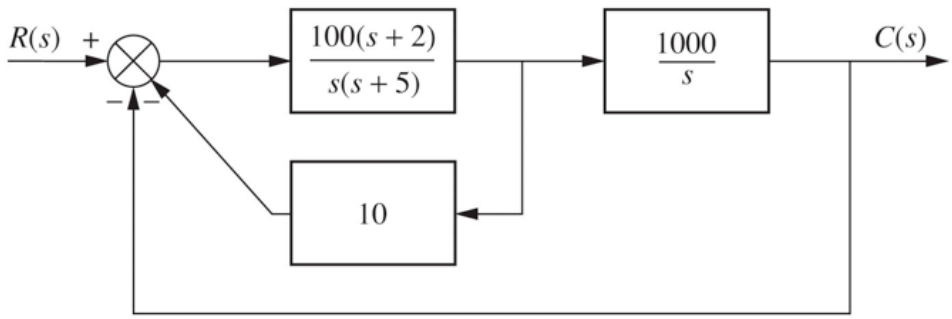

Figure 3: Block diagram for Problem 15.

Problem 38. The block diagram for this problem is shown below in Figure 4. Both reference input $R(s)$ and disturbance input $D(s)$ are unit step functions.

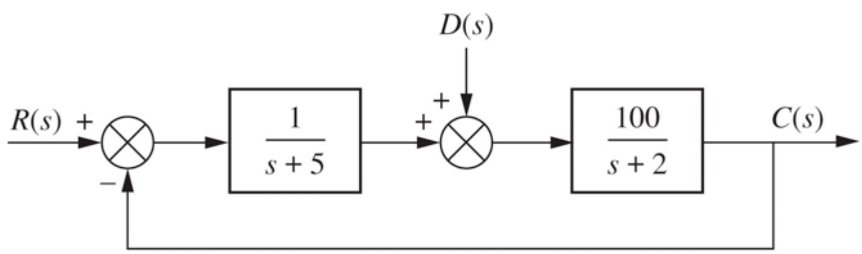

Figure 4: Block diagram for Problem 38.

Even though the text asks for the **TOTAL** steady-state error, I wanted you to find the error **SEPARATELY** due to the unit step input, and the unit step disturbance. The total error will be the sum of these two errors, but I think it's instructive to first find them separately.

a. Error due to **reference input** $R(s)$. Here you can use the **position error constant** K_p to find the error,

$$K_p = G(0) = \frac{100}{(5)(2)} = 10 \tag{15}$$

and the steady-state error is

$$\boxed{(e_{ss})_{ref} = \frac{r_o}{1+K_p} = \frac{1}{11} = 0.0909} \tag{16}$$

b. To find the error due to the **disturbance input** $D(s)$, you have to use the disturbance transfer function, which is the second part of text equation (7.60):

$$E(s) = -\frac{G_2(s)}{1+G_1(s)G_2(s)}D(s) = -\frac{100(s+5)}{(s+2)(s+5)+100} \times \frac{1}{s} \tag{17}$$

Applying the Final Value Theorem to the result of (17) yields

$$\boxed{(e_{ss})_{dist} = \lim_{s=0}\left[-\frac{(s)100(s+5)}{(s+2)(s+5)+100} \times \frac{1}{s}\right] = -\frac{500}{110} = -4.5455} \tag{18}$$

The error from the unit step disturbance is **much larger** than the error from the unit step reference input. To **reduce** this error to the disturbance we would need to increase the gain **preceding** the entry point of the disturbance.

Problem 40. The block diagram for this problem is shown in Figure 5, where I added the variable label $E(s)$:

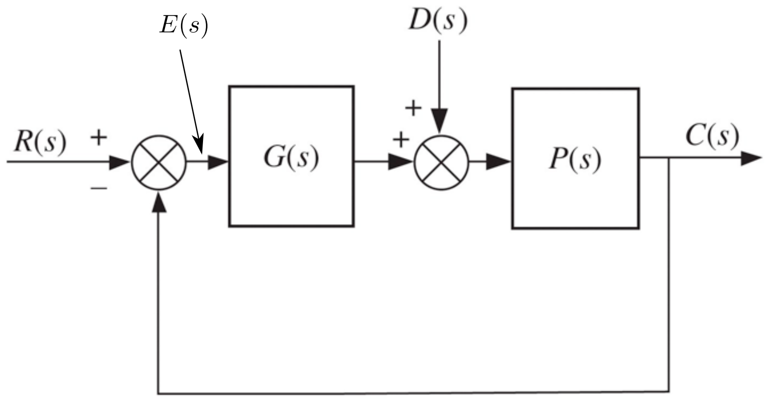

Figure 5: Block diagram for Problem 38.

We are given $G(s) = 5$ and $P(s) = \dfrac{7}{s+2}$.

The Simulink model for this system is shown in Figure 6 below:

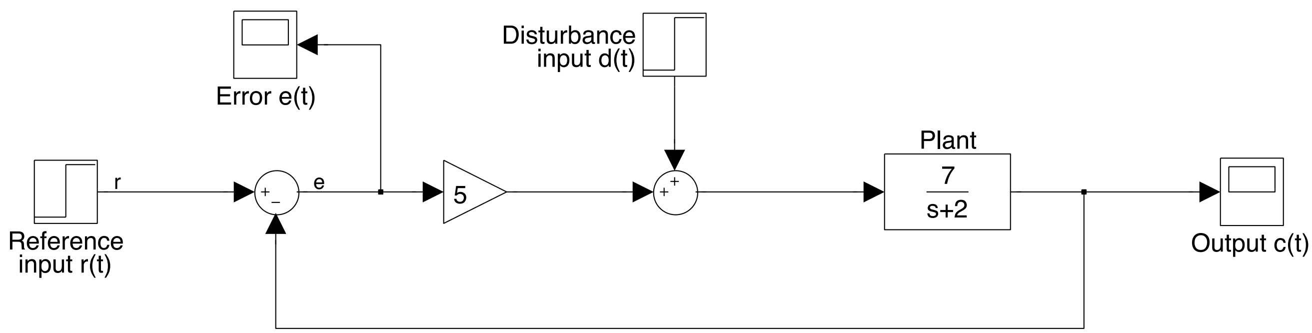

Figure 6: Simulink model for Problem 40.

a. The steady-state error due to a command (reference) input $R(s) = \dfrac{3}{s}$ can be found using the **position error constant** K_p,

$$K_p = G(0) = \frac{(5)(7)}{2} = \frac{35}{2} = 17.5 \tag{19}$$

For step input of magnitude $r_o = 3$, the steady-state error is

$$\boxed{e_{ss} = \frac{r_o}{1+K_p} = \frac{3}{1+17.5} = 0.162} \tag{20}$$

b. A screenshot of the Simulink **scope** showing the error with this reference input is shown at right in Figure 7. It's a little hard to tell if the final value of the error is really 0.162, but I'll assume it is.

FYI, you can use a Simulink "to Workspace" block and have the variable saved to your workspace. Then you can plot it using MATLAB. That's usually what I do, but I didn't bother to do it here...

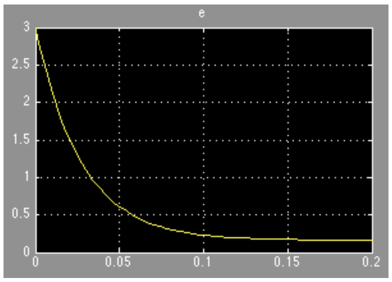

Figure 7: Error to reference input.

c. To find the error to the disturbance input **disturbance input** $D(s)$, you again have to use the disturbance transfer function, which for this problem is

$$E(s) = -\frac{P(s)}{1+P(s)G(s)}D(s) = -\frac{7}{s+37} \times -\frac{1}{s} \tag{21}$$

Applying the Final Value Theorem to the result of (21) yields

$$\boxed{(e_{ss})_{dist} = \lim_{s=0} (s) \left[-\frac{7}{s+37} \times -\frac{1}{s} \right] = \frac{7}{37} = 0.189} \tag{22}$$

d. A screenshot of the Simulink **scope** showing the error with this reference input is shown at right in Figure 8. Again, it's a little hard to tell if the final value of the error is exactly 0.189, but I'll assume it is.

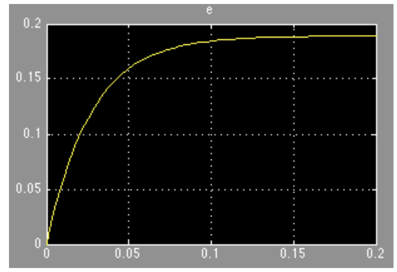

Figure 8: Error to disturbance input.

e. With both the reference input from (a) and the disturbance input from (e) acting simultaneously, the resulting error is simply the **sum** of the errors to both.

Thus we have

$$\boxed{(e_{ss})_{ref+dist} = 0.162 + 0.189 = 0.351} \tag{23}$$

f. And the Simulink response window is shown in Figure 9 below.

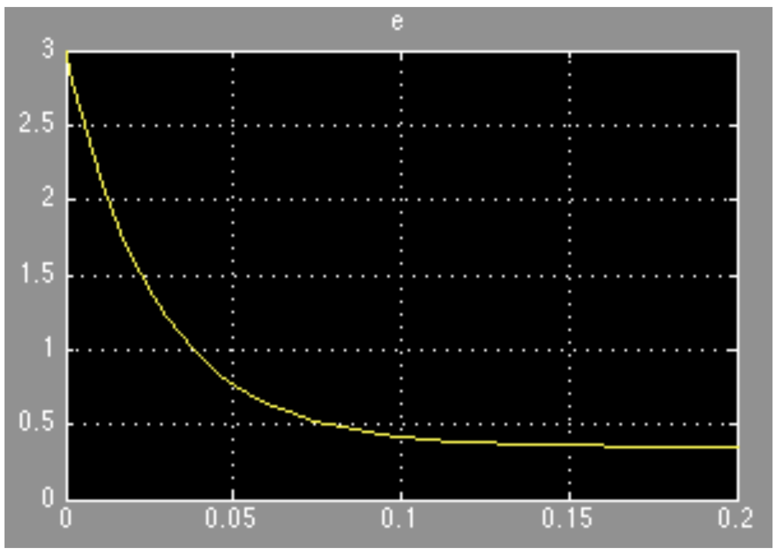

Figure 9: Error with both inputs present.

Looks llike the steady-state error in Figure 9 does approach about 0.35, so that's confirmation.

Starr Problem. This is a problem I created that deals with **SENSITIVITY ANALYSIS.**

Consider a simple **angular velocity control system**, using an amplifier and a DC motor/load. It is possible to have both an **open-loop** angular velocity control system, and a **closed-loop** system.

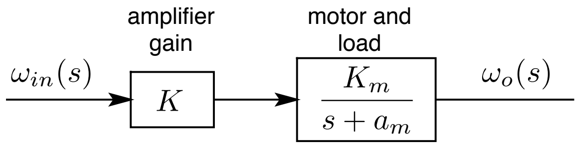

Figure 10: Open-loop angular velocity control system.

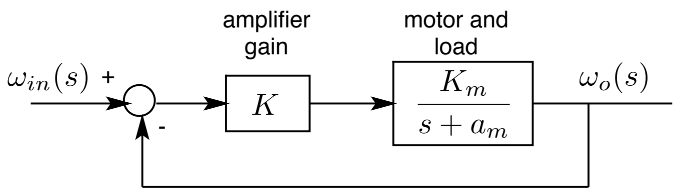

Figure 11: Closed-loop angular velocity control system.

Please do the following:

a. Find the transfer function $\frac{\omega_o(s)}{\omega_{in}(s)}$ for the open-loop system.

The transfer function for the **OPEN LOOP** system is

$$\boxed{\left[\frac{\omega_o(s)}{\omega_{in}(s)}\right]_{\text{OL}} = \frac{KK_m}{s+a_m}} \tag{24}$$

b. Find the transfer function $\frac{\omega_o(s)}{\omega_{in}(s)}$ for the closed-loop system.

The transfer function for the **CLOSED LOOP** system is

$$\boxed{\left[\frac{\omega_o(s)}{\omega_{in}(s)}\right]_{\text{CL}} = \frac{KK_m}{s+a_m+KK_m}} \tag{25}$$

c. Find the **DC gain** of the transfer function $\frac{\omega_o(s)}{\omega_{in}(s)}$ for the open-loop system.

To find the DC gain of a transfer function, just let $s = 0$, so the **DC Gain** of the transfer function for the **OPEN LOOP** system is

$$\boxed{\left[\frac{\omega_o(s)}{\omega_{in}(s)}\right]_{\text{OL DC}} = \frac{KK_m}{a_m}} \tag{26}$$

d. Find the **DC gain** of the transfer function $\frac{\omega_o(s)}{\omega_{in}(s)}$ for the closed-loop system.

Likewise, the **DC Gain** of the transfer function for the **CLOSED LOOP** system is

$$\boxed{\left[\frac{\omega_o(s)}{\omega_{in}(s)}\right]_{\text{DL DC}} = \frac{KK_m}{a_m+KK_m}} \tag{27}$$

e. Find the **sensitivity** of the DC gain of (c) to the parameter K. The sensitivity of the **open-loop DC gain** to the amplifier gain K can be found by

$$\boxed{S_{OL\ DC:K} = \frac{Ka_m}{KK_m}\frac{\partial}{\partial K}\left[\frac{KK_m}{a_m}\right] = 1} \tag{28}$$

So if the amplifier gain K increases by 10%, the open-loop DC gain **also** increases by 10%.

f. Find the **sensitivity** of the DC gain of (d) to the parameter K. The sensitivity of the **closed-loop DC gain** to the amplifier gain K can be found by

$$S_{CL\ DC:K} = \frac{K(a_m + KK_m)}{KK_m}\frac{\partial}{\partial K}\left[\frac{KK_m}{a_m + KK_m}\right] \tag{29}$$

The partial derivative of the bracketed fraction requires a little algebra...

$$S_{CL\ DC:K} = \frac{a_m + KK_m}{K_m}\ \frac{(a_m + KK_m)K_m - (KK_m)K_m}{(a_m + KK_m)^2} \tag{30}$$

The final result is

$$\boxed{S_{CL\ DC:K} = \frac{a_m + KK_m - KK_m}{a_m + KK_m} = \frac{a_m}{a_m + KK_m},\ \text{which is} < 1} \tag{31}$$

g. Which system has the **least** sensitivity of DC gain to variations in amplifier gain K?

Comparing the sensitivities of (28) and (31), you can see that the closed-loop system is less sensitive to variations in amplifier gain K. The higher the gain K, the less the sensitivity of the closed-loop system.

This is one of the advantages of closed-loop feedback systems.