Chapter 5 HW Solution

Review Questions.

1, 6. As usual, I think these are just a matter of text lookup.

1. Name the four components of a block diagram for a linear, time-invariant system. Let's see, I guess from the text Figure 5.2 there are

1. Signals (or variables)
2. Systems (*i.e.* transfer functions)
3. Summing junctions
4. Pickoff points

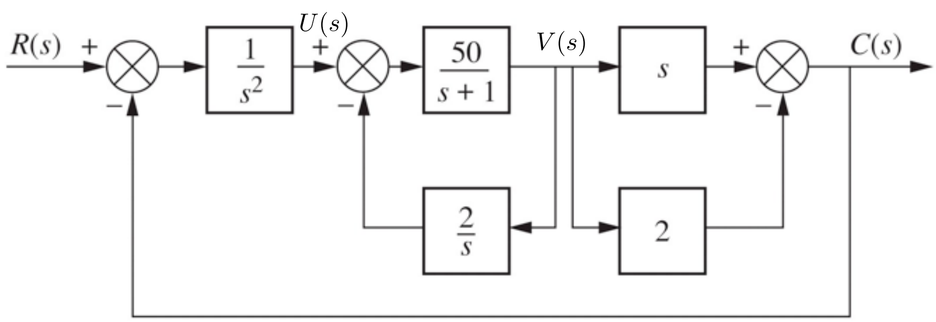
6. For a simple, second-order feedback control sysem of the type shown in Figure 5.14, describe the changes in damping ratio as the gain, K, is increased over the underdamped region.

As the gain is increased the damping ratio decreases.

Problems.

1, 11, 15, Starr Problem.

Problem 1(a). Below is my modified text Figure P5.1 with "names" for two of the internal variables in the block diagram:



First reduce the inner feedback loop:

$$\frac{V(s)}{U(s)} = \frac{50s}{s(s+1)+100} = \frac{50s}{s^2+s+100} \tag{1}$$

Next reduce the two blocks in parallel:

$$\frac{C(s)}{V(s)} = s-2 \tag{2}$$

The forward path TF is then the product of (1), (2), and $\frac{1}{s^2}$:

$$G(s) = \frac{1}{s^2} \times \frac{50s}{s^2+s+100} \times (s-2) = \frac{50(s-2)}{s(s^2+s+100)} \tag{3}$$

There is then an outer unity feedback loop around the forward path of (3),

$$\boxed{\frac{C(s)}{R(s)} = \frac{50(s-2)}{s(s^2+s+100)+50(s-2)} = \frac{50(s-2)}{s^3+s^2+150s-100}} \tag{4}$$

(b) Using MATLAB, follow basically the same procedure. I wrote a script and got the same result.

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>> prob1b

Transfer function:
      50 s - 100
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s^3 + s^2 + 150 s - 100
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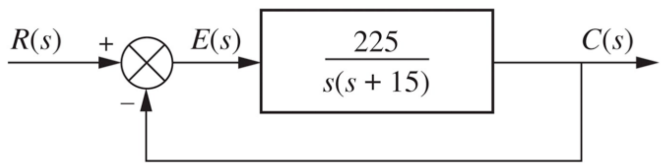
I did have to use the 'minreal' function to do the final cancellation.

Problem 11. The block diagram for this problem is shown below:



Reducing the single feedback loop, the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{225}{s(s+15)+225} = \frac{225}{s^2+15s+225} = \frac{K}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{5}$$

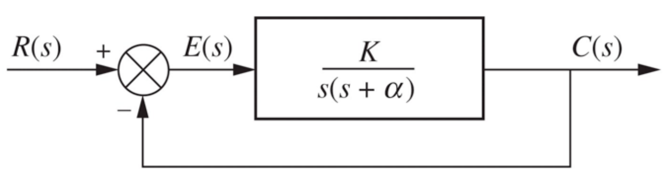
So we can find

$$\omega_n = 15 \text{ rad/s}$$
$$\zeta = \frac{15}{2(15)} = 0.5 \text{ (underdamped)}$$

Using this damping ratio and natural frequency, you get

$$\boxed{\begin{aligned} \%\,OS &= 15\% \\ 2\%\ T_s &= \frac{4}{\zeta\omega_n} = 0.53 \text{ sec} \\ T_p &= \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.24 \text{ sec} \end{aligned}}$$

Problem 15. The block diagram for this system is shown below:



Reducing the feedback loop, we get

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+\alpha)+K} = \frac{K}{s^2+\alpha s+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{6}$$

So we have

$$\omega_n = \sqrt{K}, \quad \zeta = \frac{\alpha}{2\sqrt{K}} \tag{7}$$

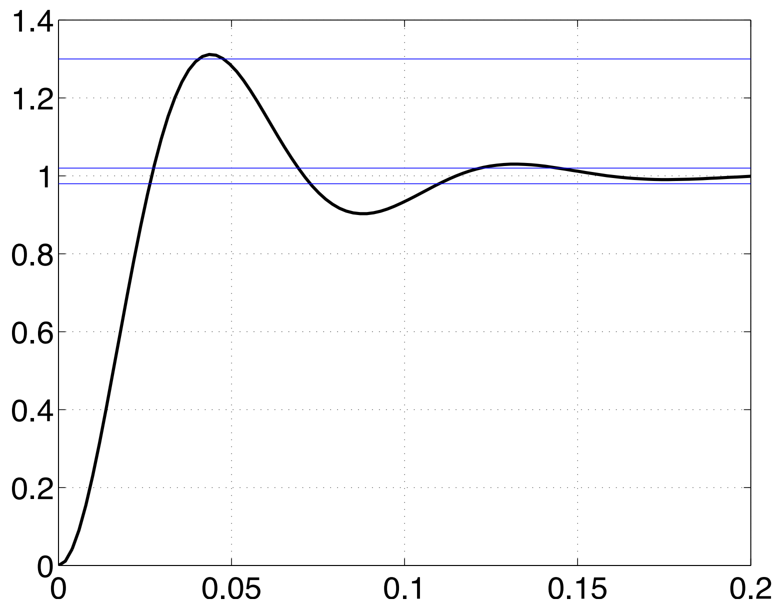
Design Specifications:

$$30\%\ \text{OS} \implies \zeta = 0.35$$
$$T_s = \frac{4}{\zeta\omega_n} = 0.15 \implies \omega_n = 76\ \text{rad/s}$$

Thus

$$\boxed{K = \omega_n^2 = 5805, \quad \alpha = 2\zeta\sqrt{K} = 53} \tag{8}$$

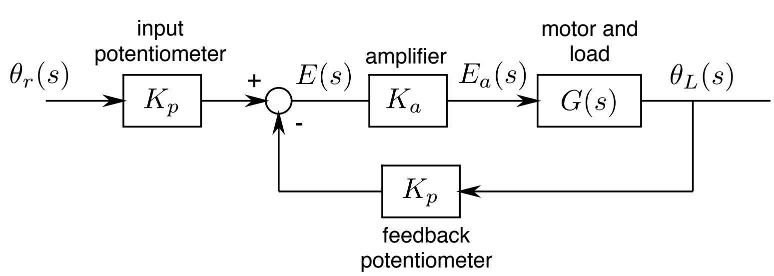
Although not required, the Problem 15 step response is shown below; the 30% overshoot and 2% settling time are shown on the plot. Pretty good agreement.



Starr DC Motor Problem. Consider the **DC Motor and Load** that I assigned for Chapters 3 and 4. This motor will be driven by an amplifier with voltage gain K_a.

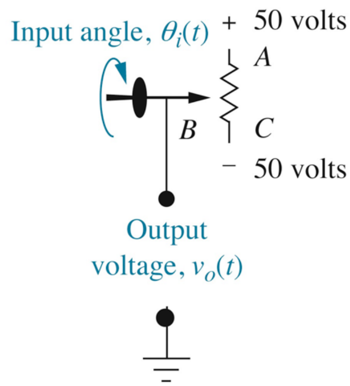
The position of the load will be measured by the potentiometer described in Chapter 1 HW Problem 1: a 10-turn potentiometer with ±50V across it. We found the "transfer function" (it's just a constant gain) back in the Chapter 1 HW. Also consider that the reference input is the desired angle of another of the same potentiometers.

A block diagram of the system is shown below:



Let the transfer function of the "motor and load" be the $G(s)$ of the Chapter 4 HW problem, with load position θ_L in rad, and motor voltage e_a in V. Both the "input potentiometer" and "feedback potentiometer" are identical, and have transfer function (gain) of K_p V/rad in accordance with Chapter 1 Problem 1.

(a) Find the closed-loop transfer function $\dfrac{\theta_L(s)}{\theta_r(s)}$. From the Chapter 1 HW, a diagram of the (10-turn) potentiometers is:



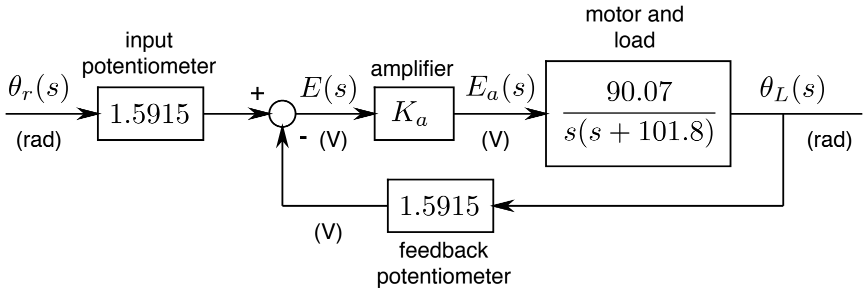
From the Chapter 1 HW, the "transfer function" K_p of the input & feedback potentiometer (they are identical) is

$$K_p = \frac{V(s)}{\theta(s)} = \frac{50-(-50)}{(10)(2\pi)} = \frac{100}{20\pi} = 1.5915 \;\; \frac{\text{V}}{\text{rad}} \tag{9}$$

From the Chapter 4 HW, the transfer function of the Motor+Load is:

$$\frac{\theta_L(s)}{E_a(s)} = \frac{\dfrac{K_t}{nR_aJ_t}}{s\left(s+\dfrac{b_t}{J_t}\right)} = \frac{90.07}{s(s+101.8)} \;\; \frac{\text{rad}}{\text{V}} \tag{10}$$

The system block diagram is then shown below, with all numerical blocks populated. Note that the units of all variables are shown in parentheses.



This is a simple feedback loop with an input transfer function preceding it. The closed-loop transfer function is

$$\boxed{\frac{\theta_L(s)}{\theta_r(s)} = K_p \times \frac{90.07K_a}{s(s+101.8)+90.07K_aK_p} = \frac{90.07K_aK_p}{s^2+101.8s+90.07K_aK_p} = \frac{143.3K_a}{s^2+101.8s+143.3K_a} \;\; \frac{\text{rad}}{\text{rad}}} \tag{11}$$

(b) Find the value of amplifier gain K_a to yield critical damping.

From the closed-loop system TF of (11), we have

$$\omega_n = \sqrt{143.3K_a} \implies \omega_n \approx 12\sqrt{K_a}, \tag{12}$$

thus

$$\boxed{\zeta = \frac{101.8}{24\sqrt{K_a}} = 1 \implies K_a = \left(\frac{101.8}{24}\right)^2 = 18 \;\; \frac{\text{V}}{\text{V}}} \tag{13}$$

(c) What is the DC gain of the closed-loop system?

Denoting the closed-loop transfer function of (11) by $T(s)$, we have

$$\boxed{\text{DC Gain} = \left[\frac{\theta_L}{\theta_r}\right]_{ss} = T(0) = 1.} \tag{14}$$

So **1 turn** of the input potentiometer will (eventually) cause a motion of **1 turn** of the load position. That's what DC Gain means...

(d) Plot the step response of the system to a 1-turn step input.

Since the reference input is in units of radians, the input is a step of magnitude 2π. The output is shown below:

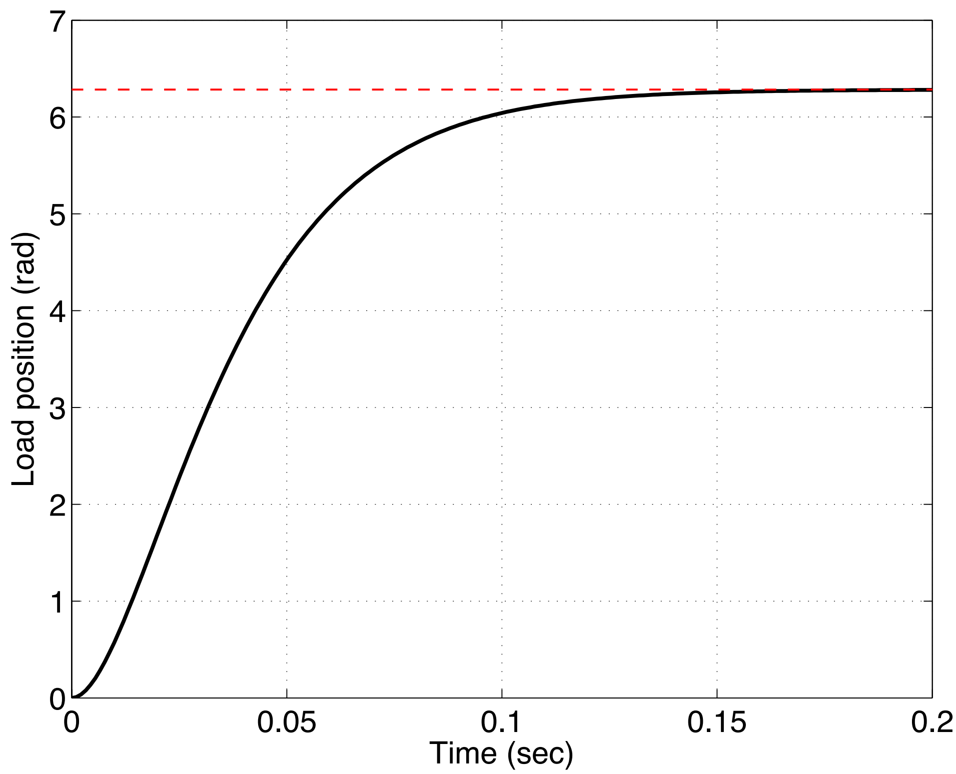
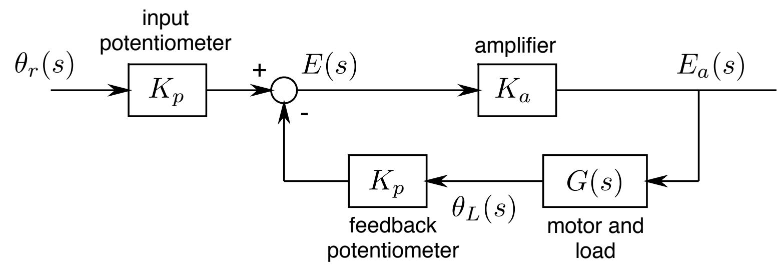


Figure 1: Response of load to 1-turn reference step input.

The red dashed line shows the final value (DC Gain = 1). Critical damping provides the fastest possible response to a step input with no overshoot.

(e) Plot motor voltage e_a in response to the input of (d). You will have to re-draw the block diagram showing motor voltage as the output. The "motor and load" will now be in the feedback path, and you will have to find the transfer function $E_a(s)/\theta_{r(s)}$. It should have the same denominator as in (a), but the numerator will be different.

Re-drawing the block diagram (as shown in class) to expose E_a as the output, we get



Reducing this TF in MATLAB, I got

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28.65 s^2 + 2916 s
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s^2 + 101.8 s + 2580
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Expressing this TF in a factored form, we get

$$\frac{E_a(s)}{\theta_r(s)} = \frac{28.6s(s + 101.8)}{(s + 54.14)(s + 47.66)} \tag{15}$$

Notice that the two poles are not quite equal—this was probably due to my rounding-off of intermediate calculations. Well, none of this stuff is really that precise!

The step response is shown below.

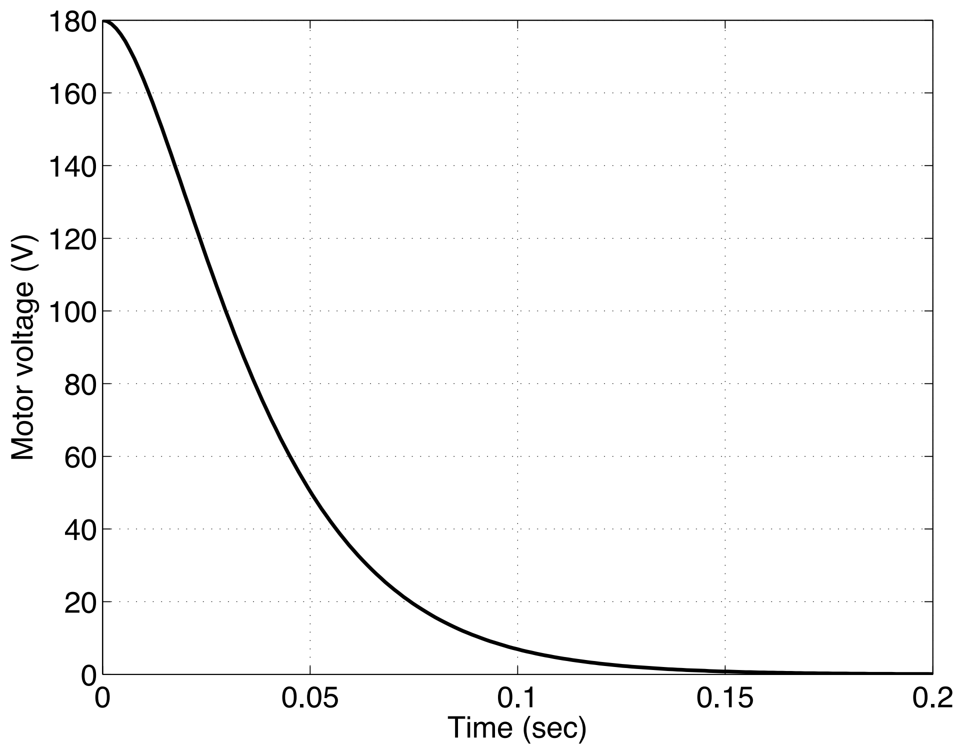

Figure 2: Voltage applied to motor in response to 1-turn reference step input.

So, in response to a one-turn step input, the motor voltage jumps immediately to 180 volts!! Do you think the motor can handle that? Without motor specifications there's no way to tell, but it is likely that the motor would **saturate**, and that it wouldn't move quite as fast as you predicted.

So a one-turn step input is too large of a command. As I've said, a **step input** is more for analytical test purposes, and is often not something you'd actually do in a lab with real hardware. You'd apply a much "gentler" input...