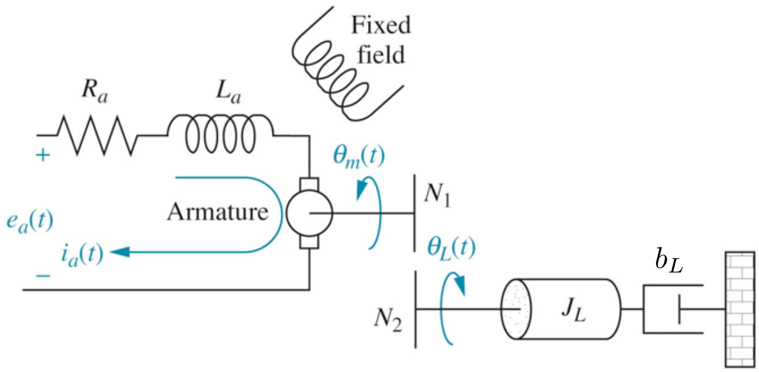
Chapter 3 HW Solution

Problem 19. Consider a DC motor driving a load through a gear train:



The specifications for the Pittman 7214 DC motor (taken from the manufacturer's data sheet) are:

$$
\begin{aligned}
J_m &= 1.54 * 10^{-6} \text{ kg-m}^2 \quad \text{(not shown in figure)} \\
b_m &= 2.7e-3 \ \frac{\text{mN-m}}{\text{rad/s}} \quad \text{(not shown in figure)} \\
L_a &= 0.69 \text{ mH} \\
R_a &= 1.53\ \Omega \\
K_t &= 22.3 \ \frac{\text{mN-m}}{\text{A}} \\
K_b &= 0.0223 \ \frac{\text{V}}{\text{rad/s}}
\end{aligned}
$$

The load inertia is a disk made of aluminum with the following dimensions:

$$
\begin{aligned}
\text{radius } r &= 100 \text{ mm} \\
\text{thickness } t &= 10 \text{ mm} \\
\text{viscous damping } b_L &= 4 \ \frac{\text{mN-m}}{\text{rad/s}} \\
\text{gear ratio } n &= \frac{N_2}{N_1} = 50
\end{aligned}
$$

(a) As presented in class, the three equations that model an armature voltage-controlled DC motor are (with back EMF $v_b = K_b\omega_m$):

$$e_a - i_a R_a - L_a \frac{di_a}{dt} - K_b \omega_m = 0 \tag{1}$$

$$T_m = K_t i_a \tag{2}$$

$$T_m - b_m \omega_m = J_m \dot{\omega}_m \tag{3}$$

The three state variables are $x_1 = i_a$, $x_2 = \theta_m$, $x_3 = \omega_m$, so the state vector is

$$\mathbf{x} = \begin{bmatrix} i_a \\ \theta_m \\ \omega_m \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4}$$

State Equations. From (1) we can find the first state equation:

$$\frac{di_a}{dt} = \frac{1}{L_a}\left(e_a - i_a R_a - K_b \omega_m\right) \tag{5}$$

The second state equation is a simple derivative:

$$\frac{d\theta_m}{dt} = \omega_m \tag{6}$$

The third state equation comes from (3) with (2) substituted in:

$$\frac{d\omega_m}{dt} = \frac{1}{J_m}\left(K_t i_a - b_m \omega_m\right) \tag{7}$$

Substituting for the three state variables x_i and input $u = e_a$, we get the three state equations as:

$$\boxed{\begin{aligned} \dot{x}_1 &= -\frac{R_a}{L_a}x_1 - \frac{K_b}{L_a}x_3 + \frac{1}{L_a}u && (8)\\ \dot{x}_2 &= x_3 && (9)\\ \dot{x}_3 &= \frac{K_t}{J_m}x_1 - \frac{b_m}{J_m}x_3 && (10) \end{aligned}}$$

From (8)–(10) we can find system matrix $\mathbf{A}$ and input matrix $\mathbf{B}$ as

$$\boxed{\mathbf{A} = \begin{bmatrix} -R_a/L_a & 0 & -K_b/L_a \\ 0 & 0 & 1 \\ K_t/J_m & 0 & -b_m/J_m \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix}} \tag{11}$$

Output Equations. The output matrix $\mathbf{C}$ must produce three outputs: current i_a in (A), motor displacement θ_m in (rev), and motor velocity ω_m in (rpm). So matrix $\mathbf{C}$ will contain the appropriate unit conversions. What we need is

$$\boxed{\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/(2\pi) & 0 \\ 0 & 0 & 60/(2\pi) \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \tag{12}$$

where "feed-thru" matrix $\mathbf{D}$ is zero for this physical system.

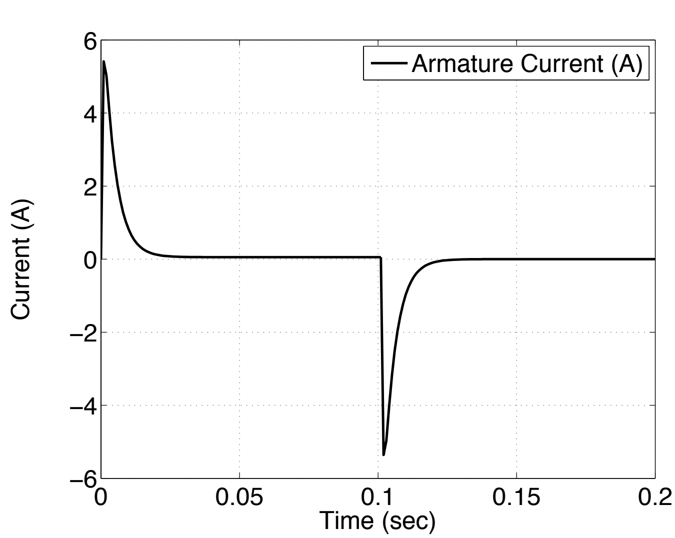
Simulation. Using a similar MATLAB script to that on the website, I simulated the motor to the voltage pulse using the MATLAB `lsim()` function.

The three plots requested,

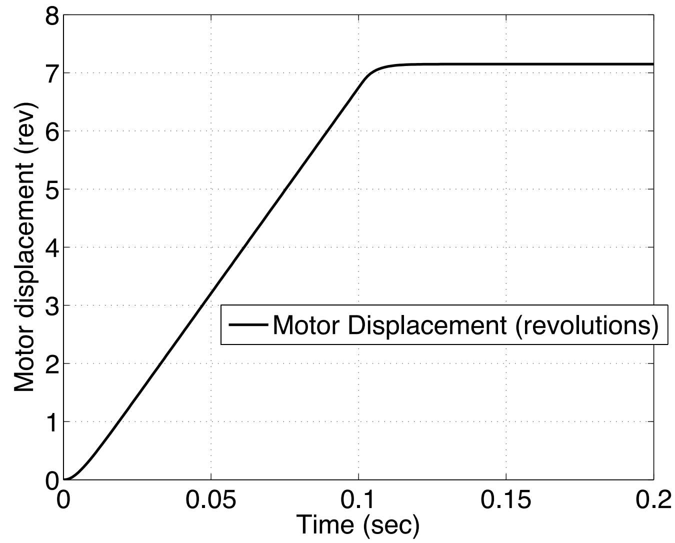
- Motor armature current i_a in units of (V),
- Motor displacement (angular rotation) θ_m in units of (rev),
- Motor velocity ω_m in units of (rpm),

are all shown on the next page. Inspection of the data shows that:

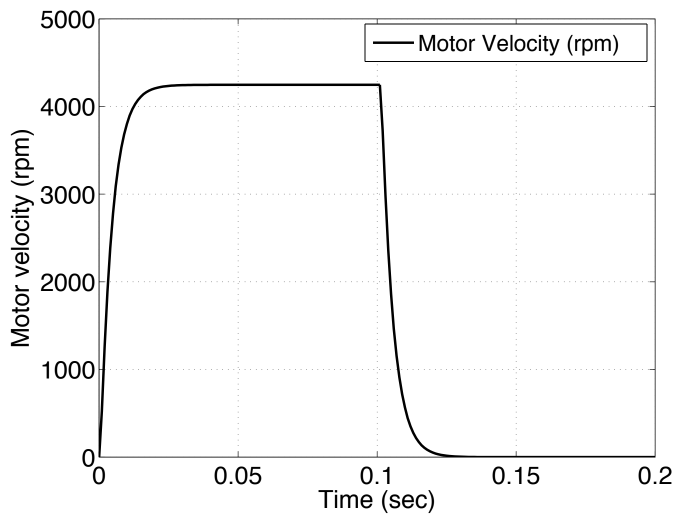
$$\boxed{\begin{aligned} \text{Maximum armature current} &= 5.4129\ \text{A} \\ \text{The motor rotated } 7.149 &\text{ revolutions} \end{aligned}}$$



(a) Armature current.



(b) Motor displacement (rev) vs time.



(c) Motor velocity (rpm) vs time.

Figure 1: Motor response to voltage pulse.

Note that in Figure 1(a) the current immediately "spikes" up to begin supplying torque, then drops off as the motor accelerates to a constant speed, as seen in Figure 1(c). The motor rotation of 7.149 revolutions appears in Figure 1(b). Also, the maximum speed of the motor is 4,247 rpm. That's why we use **gear trains**—these motors spin *FAST!*

(b) Repeat part (a) but now **neglect** the armature inductance. You will now have only two state variables: $x_1 = \theta_m$ and $x_2 = \omega_m$. You will also now have only two outputs. **NOTE:** Since armature current i_a is no longer a state variable, you will have to figure out some other way of computing it to make the plot (*Hint:* look at the armature voltage equation).

State Equations. Neglecting L_a, the three motor equations are the following:

$$e_a - i_a R_a - K_b \omega_m = 0 \tag{13}$$

$$T_m = K_t i_a \tag{14}$$

$$T_m - b_m \omega_m = J_m \dot{\omega}_m \tag{15}$$

Since motor current i_a is no longer a state variable, it can be eliminated. Do this by solving (13) for current i_a:

$$i_a = \frac{1}{R_a}\left(e_a - K_b\omega_m\right) \tag{16}$$

Substitute from (16) for i_a into (14), and you get

$$T_m = \frac{K_t}{R_a}\left(e_a - K_b\omega_m\right) \tag{17}$$

Now substitute T_m from (17) into (15), rearrange, and you get

$$\dot{\omega}_m = -\frac{1}{J_m}\underbrace{\left(\frac{K_tK_b}{R_a} + b_m\right)}_{b_{eq}}\omega_m + \frac{K_t}{R_a}e_a \tag{18}$$

where the term b_{eq} is defined to include both the motor viscous friction and the "electronic" back EMF damping. This just simplifies things.

Now equation (18) is actually the second state equation. Remember the state variables are now $x_1 = \theta_m$ and $x_2 = \omega_m$. So using "standard" notation the two state equations for the motor (neglecting L_a) are:

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b_{eq}}{J_m}x_2 + \frac{K_t}{J_mR_a}u \end{aligned}} \begin{matrix}(19)\\(20)\end{matrix}$$

where again $u = e_a$ is the motor input.

From (19)-(20) the simplified motor system and input matrices are

$$\boxed{\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -b_{eq}/J_m \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ K_t/(J_mR_a) \end{bmatrix}} \tag{21}$$

Output Equations. Examination of (16) shows that armature current i_a can be written as

$$i_a = \frac{-K_b}{R_a}x_2 + \frac{1}{R_a}u, \tag{22}$$

so motor current i_a can be computed as an output of interest using the proper $\mathbf{C}$ and $\mathbf{D}$ matrices.

The output matrix $\mathbf{C}$ is now 3×2, and performs the current computation of (22) plus the same unit conversions:

$$\boxed{\mathbf{C} = \begin{bmatrix} 0 & -K_b/R_a \\ 1/(2\pi) & 0 \\ 0 & 60/(2\pi) \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1/R_a \\ 0 \\ 0 \end{bmatrix}} \tag{23}$$

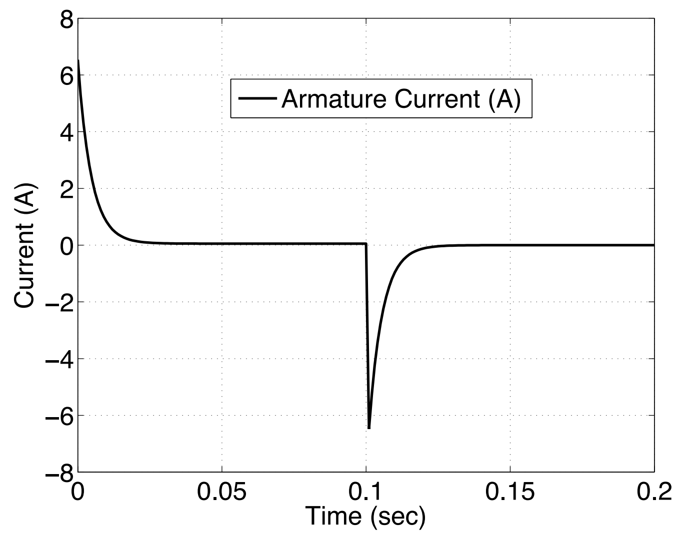
Here is a case in which we have a "physical system," but the $\mathbf{D}$ matrix is non-zero. It *CAN* happen...

Simulation. Using the same MATLAB script as before (with the different matrices), I simulated the motor to the voltage pulse using the MATLAB `lsim()` function.

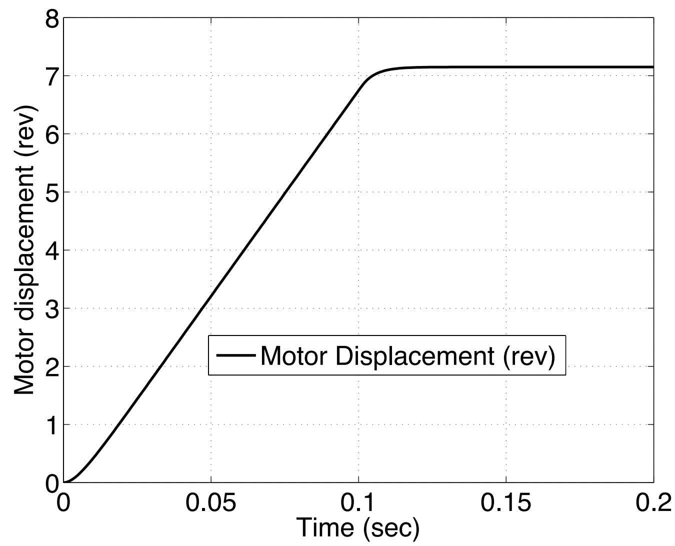
The same three plots,

- Motor armature current i_a in units of (V),
- Motor displacement (angular rotation) θ_m in units of (rev),
- Motor velocity ω_m in units of (rpm),

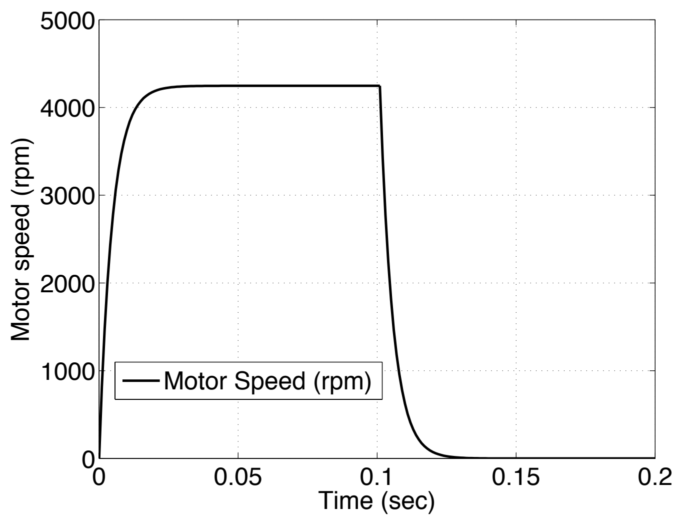
are all shown below. Inspection of the data with neglected L_a shows that:



(a) Armature current.



(b) Motor displacement (rev) vs time.



(c) Motor velocity (rpm) vs time.

Figure 2: Motor response with armature inductance L_a neglected.

$$\boxed{\begin{array}{r}\text{Maximum armature current} = 6.5359\text{ A}\\ \text{The motor rotated 7.149 revolutions}\end{array}}$$

Is there much difference in the response of the motor compared to (a)? NO! The armature current is a little larger (neglecting the armature inductance changes the current dynamics slightly), but the mechanical behavior of the motor (position and velocity) is almost **EXACTLY** the same. That is why the armature inductance is commonly neglected...

(c) Now add the dynamics of the load (load inertia and damping) to the model of part (b), but ignore the inertia of the gears (it was not given anyway!). Perform the same simulation as in (a). Plot motor current (A), load velocity (degrees/sec), and load displacement (degrees) *vs* time.

Basically only two things need to be changed from the analysis of part (b):

- The load inertia and damping need to be added to the that of the motor
- The **C** output matrix needs to produce load velocity (deg/s) and load displacement (deg)

Load Inertia and Damping. The load is an aluminum disk of thickness $t = 10$ mm and radius $r = 100$ mm. The mass moment of inertia of a cylinder about its center is

$$J_L = \frac{1}{2} m r^2 \;\; \text{kg-m}^2 \tag{24}$$

The mass is not given, but the density of Al is $\rho = 2.7$ g/cm^3. The mass can be calculated as 0.8482 kg, and the mass moment of inertia of the load is

$$J_L = 0.004241 \;\; \text{kg-m}^2 \tag{25}$$

Since the motor drives the load through a gear train of ratio n, the total inertia and damping of the system as "felt" by the motor are:

$$J_t = J_m + \frac{J_L}{n^2} = 3.2365\text{e-06 kg-m}^2 \tag{26}$$

$$b_t = b_{eq} + \frac{b_L}{n^2} = 3.2933\text{e-04 N-m-s/rad} \tag{27}$$

Note that the mass moment of inertia just about doubled when the load was added—not **that much** of an increase.

Output Matrix. To yield load displacement (deg) and load velocity (deg/s), output matrix **C** is

$$\boxed{\mathbf{C} = \begin{bmatrix} 0 & -K_b/R_a \\ 180/(n\pi) & 0 \\ 0 & 180/(n\pi) \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1/R_a \\ 0 \\ 0 \end{bmatrix}} \tag{28}$$

Armature current i_a is computed as y_1 in the same manner as in part (b)

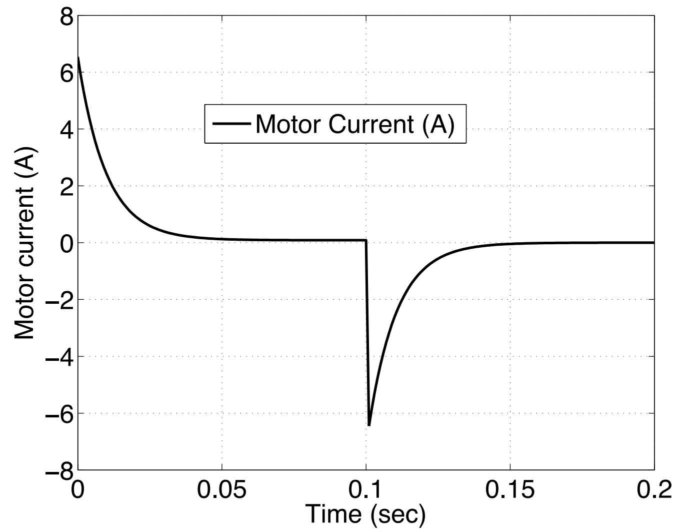
Simulation. Using the same MATLAB script as before (with appropriate changes), I simulated the motor to the voltage pulse using the MATLAB `lsim()` function.

The same plots are on the next page. Inspection of the data with neglected L_a shows that:

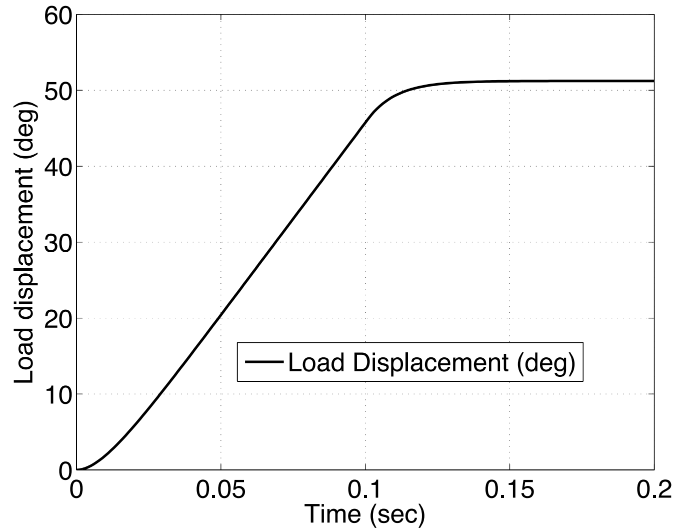
$$\boxed{\begin{aligned} \text{Maximum armature current} &= 6.5359 \text{ A} \\ \text{Total load rotation} &= 51.2223^\circ \end{aligned}}$$

It is interesting that the maximum load current **WITH** the load is exactly the same as **WITHOUT** the load. This doesn't really make sense. The reason why this is the case is that in these two models we **NEGLECTED** armature inductance, which effectively removed the modeling of armature current behavior. The "derived" armature current didn't accurately reflect the effect of the load.

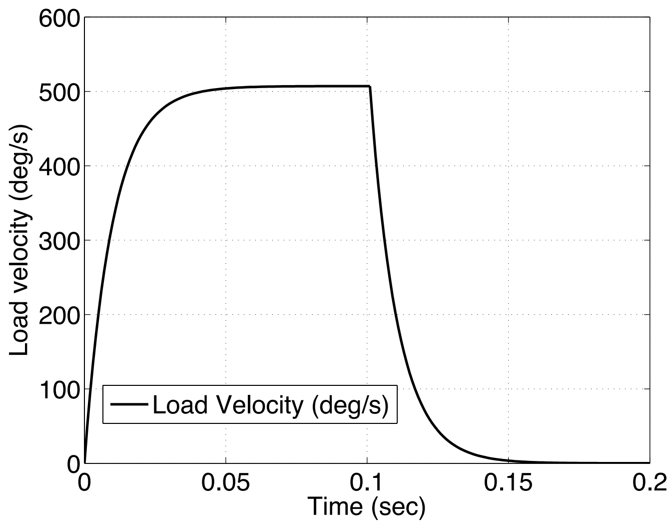
Examination of the "shape" of the response of Figure 3(c) shows the somewhat **slower response** of the system with the load present.



(a) Armature current.



(b) Load displacement (deg) vs time.



(c) Load velocity (deg/s) vs time.

Figure 3: Motor *and Load* response with armature inductance L_a neglected.

This was a realistic problem of modeling a *REAL* DC motor driving a rotary load through a gear train. Neglecting armature inertia was a good assumption. Components like these are very commonly used for "incremental motion" systems in industry.