ME 380 Chapter 2 HW Solution

Review Questions.

1. What mathematical model permits easy interconnection of physical sysems? The transfer function model.

3. What transformation urns the solution of differential equations into algebraic manipulation? The Laplace transformation.

4. Define the transfer function. The transfer function is the ratio of the Laplace transform of the output over the Laplace transform of the input.

5. What assumption is made concerning initial conditions when dealing with transfer functions? We assume that the initial conditions are **zero**.

9. What function do gears perform? Typically a gear train is used to **reduce** speed and **increase** torque. In some cases the reverse is true, but the majority are speed reducers (*e.g.* automotive transmissions).

Problems.

Problem 8a. For the following transfer function, write the corresponding differential equation.

$$\frac{X(s)}{F(s)} = \frac{7}{s^2 + 5s + 10} \tag{1}$$

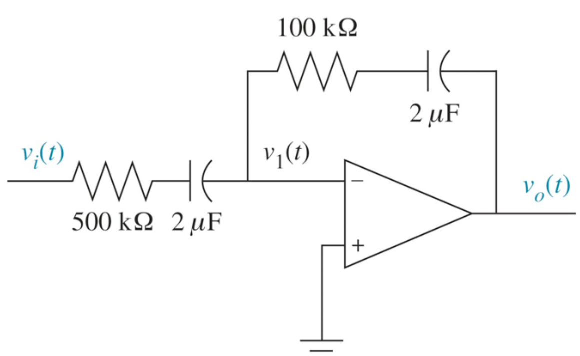
If you cross-multiply, you get

$$s^2X(s) + 5sX(s) + 10X(s) = 7F(s) \tag{2}$$

Since multiplication by "s" corresponds to time differentiation, the result is

$$\boxed{\ddot{x}(t) + 5\dot{x}(t) + 10x(t) = 7f(t)} \tag{3}$$

Problem 21a. Find the transfer function $G(s) = V_o(s)/V_i(s)$ of the op-amp circuit shown below.



As I stated in class, an "inverting amplifier" circuit of this type with input impedance $Z_i(s)$ and feedback impedance $Z_f(s)$ will have TF

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_i(s)}, \tag{4}$$

Substituting the values for resistance and capacitance, we get

$$Z_i(s) = R_i + \frac{1}{C_i s} = \frac{s+1}{(2e-6)s} \tag{5}$$

$$Z_f(s) = R_f + \frac{1}{C_f s} = \frac{0.2s+1}{(2e-6)s} \tag{6}$$

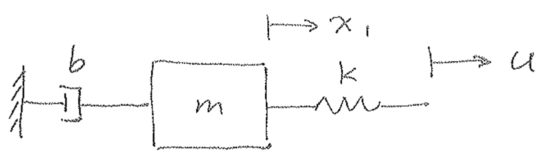
Therefore the final TF is

$$\boxed{G(s) = -\frac{Z_f(s)}{Z_i(s)} = -\frac{0.2s+1}{s+1} = -\frac{0.2(s+5)}{s+1}} \tag{7}$$

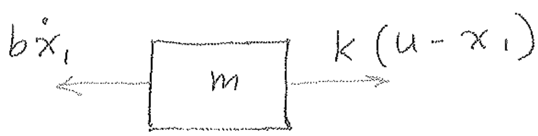
BTW, from its frequency response properties, this thing is called a **LAG** network (kind of a psseudo-integrator). It can be used to improve the steady-state response of a feedback control system.

Problem 23. I changed the input to this problem: instead of force $f(t)$ applied to the end of the spring, the input is **displacement** $u(t)$ of the free end of the spring (directed positive to the right like displacement x_1). This is much more realistic and easier to analyze.

Thus the problem sketch is like this, where I've define parameters m, b, and k so you can first analyze the problem symbolically (always a good idea):



My free-body diagram is



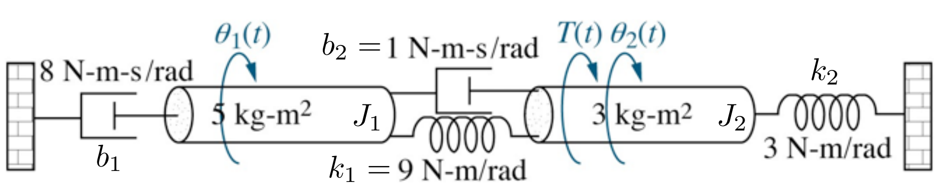
Applying Newton's 2nd Law ($\Sigma f = m\ddot{x}$) we get the following equation of motion:

$$-b\dot{x}_1 + k(u - x_1) = m\ddot{x}_1 \implies m\ddot{x}_1 + b\dot{x}_1 + kx_1 = ku \tag{8}$$

Laplace transforming (8) and rearranging to get a TF yields the result:

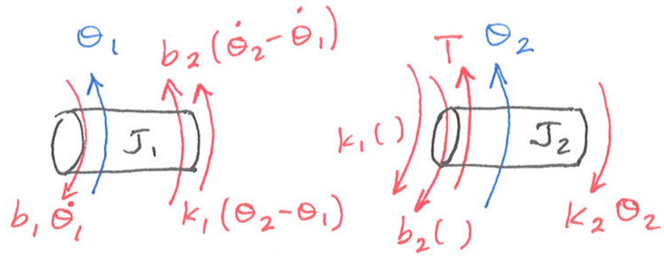
$$\boxed{\frac{X_1(s)}{U(s)} = \frac{k}{ms^2+bs+k} = \frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} = \frac{1}{s^2+0.8s+1}\ \frac{\text{m}}{\text{m}}} \tag{9}$$

Problem 30a. My version of this problem sketch is shown below. We're supposed to find the equations of motion (one for each rotary inertia).



I told you to use inertia, damping, and stiffness parameters $J_1, J_2, b_1, b_2, k_1, k_2$ in your analysis, since it's *ALWAYS* preferable to use symbolic parameters (rather than numeric).

For purposes of visualizing the torque directions for the free-body diagram, I assumed that $\theta_2 > \theta_1$, and $\dot{\theta}_2 > \dot{\theta}_1$ (you could assume the opposite). My free-body diagram is shown below:



As I said in the "hints", the two moments on J_2 labeled $k_1()$ and $b_2()$ are just $k_1(\theta_2 - \theta_1)$ and $b_2(\dot{\theta}_2 - \dot{\theta}_1)$.

To find the equations of motion (EOM), just apply $\Sigma T = J\ddot{\theta}$ for each rotary inertia.

The EOMs for both inertias are:

$$J_1\ddot{\theta}_1 + (b_1 + b_2)\,\dot{\theta}_1 + k_1\theta_1 - b_2\dot{\theta}_2 - k_1\theta_2 = 0 \tag{10}$$

$$J_2\ddot{\theta}_2 + b_2\dot{\theta}_2 + (k_1 + k_2)\,\theta_2 - b_2\dot{\theta}_1 - k_1\theta_1 = T \tag{11}$$

I would stop there, but the author gives us (phony) values for all these parameters, so I suppose I should substitute those:

$$\begin{aligned} b_1 &= 8 \text{ N-m-s} \\ b_2 &= 1 \text{ N-m-s} \\ k_1 &= 9 \text{ N-m/rad} \\ k_2 &= 3 \text{ N-m/rad} \\ J_1 &= 5 \text{ kg-m}^2 \\ J_2 &= 3 \text{ kg-m}^2 \end{aligned}$$

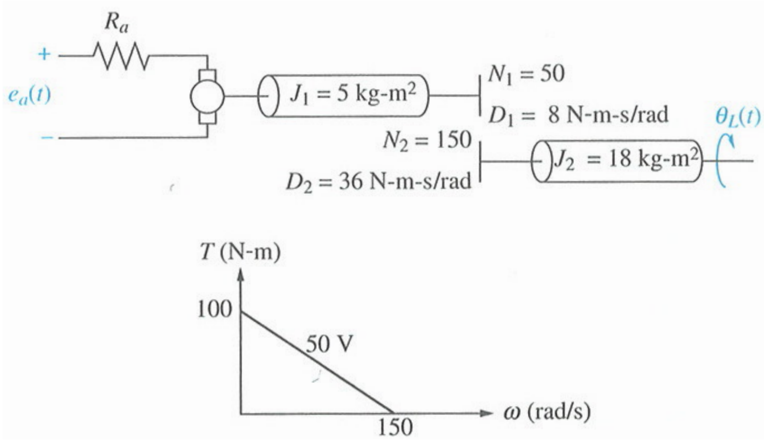
The EOMs become

$$5\ddot{\theta}_1 + 9\dot{\theta}_1 + 9\theta_1 - \dot{\theta}_2 - 9\theta_2 = 0 \tag{12}$$

$$3\ddot{\theta}_2 + \dot{\theta}_2 + 12\theta_2 - \dot{\theta}_1 - 9\theta_1 = T \tag{13}$$

Comparing (12–13) to (10–11) you can see that once you substitute numerical values, you lose track of the contribution of the parameters. However, with numerical parameters one can now perform numerical simulations (which we will do using MATLAB).

Problem 42. This is a DC motor driving a load inertia *via* a gear train.



In keeping with my classroom analysis, I used the following terminology:

$$
\begin{aligned}
J_m &= J_1 = \text{motor inertia} \\
b_m &= D_1 = \text{motor viscous friction (damping)} \\
J_L &= J_2 = \text{load inertia} \\
b_L &= D_2 = \text{load viscous friction} \\
b_{eq} &= \text{combined motor inertia and back EMF} \\
n &= \frac{N_2}{N_1} = \text{gear ratio}
\end{aligned}
$$

The total inertia the motor "feels" is J_t, and is given by

$$J_t = J_m + \frac{J_L}{n^2} \tag{14}$$

Likewise the total viscous friction the motor feels is

$$b_t = b_{eq} + \frac{b_L}{n^2} \tag{15}$$

where as you recall the "equivalent" damping coefficient is given by

$$b_{eq} = b_m + \frac{K_t K_b}{R_a} \tag{16}$$

Now, the transfer function from applied motor armature voltage $E_a(s)$ to *motor* position $\theta_m(s)$ is given by

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_t)}{s\left(s + \frac{b_t}{J_t}\right)} \tag{17}$$

where we have used the total inertia and damping (motor + load).

Given Parameters. The following parameters are given in the problem statement (using my terminology):

$$J_m, J_L, b_m, b_L, n$$

Parameters K_t, K_b, R_a must be found from the given speed-torque curve. Remember that $K_t = K_b$ in equivalent units.

Speed-Torque Equation. The equation for motor torque in terms of voltage and speed is:

$$T_m = \frac{K_t}{R_a}\left(e_a - K_b \omega_m\right) \tag{18}$$

At **stall torque** of 100 N-m, at the given voltage of $e_a = 50$ V:

$$T_{\text{stall}} = \frac{K_t}{R_a} e_a = \frac{K_t}{R_a}(50) = 100 \implies \frac{K_t}{R_a} = 2 \tag{19}$$

At **no-load speed** of 150 rad/s, at the given voltage of $e_a = 50$ V:

$$\omega_{\text{nl}} = \frac{e_a}{K_b} = \frac{50}{K_b} = 150 \implies K_b = \frac{50}{150} = \frac{1}{3}\,\frac{\text{V-s}}{\text{rad}} \tag{20}$$

Since $K_t = K_b$ we also know that

$$K_t = \frac{1}{3}\,\frac{\text{N-m}}{\text{A}} \tag{21}$$

Finally, remember that you have to use the gear ratio n to get the final result in terms of load rotation angle θ_L, where $\theta_m = n\theta_L$.

Substituting in all parameters, the final result is

$$\boxed{\frac{\theta_L(s)}{E_a(s)} = \frac{K_t/(nR_aJ_t)}{s\left(s + \frac{b_t}{J_t}\right)} = \frac{0.0952}{s(s+1.81)}\,\frac{\text{rad}}{\text{V}}} \tag{22}$$