Kinematics

1 Introduction

From previous ME 314 evaluations, the "least favorite" portion of the material was **kinematics.** In particular, students felt that we should spend less time on manual calculation of velocities and accelerations, and more time on gears, cams, balancing, *etc.*

To that end, I have decided **NOT** to follow the text treatment of analytical velocity and acceleration analysis, because I consider it to be poorly presented and very confusing. This semester I *really* wanted to try to follow the text, but...

Also, we will not be performing "graphical" kinematic analysis, which is covered in the book. We will, instead, be using ADAMS for computer-aided mechanism analysis.

Nevertheless, you *must* be exposed to analytical methods of solving for mechanism kinematics. Therefore, I am developing a very brief set of notes based on the best presentation of dynamics that have ever seen, the textbook

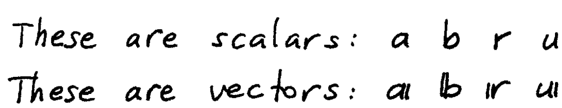
Thomas R. Kane, <u>Dynamics</u>, Holt, Rinehart, and Winston, Inc., 1968.

This is the textbook used in the Applied Mechanics 221, 222, 223 graduate course series at Stanford University during my tenure there as an M.S. and Ph.D. student. Thomas R. Kane is legendary in the dynamics field, and his concise presentation serves as an enduring example of how to write a textbook.

2 Notation and Other Preliminaries

2.1 Notation

We will be dealing with both *scalar* and *vector* quantities, which you often write by hand. I **would like you to use a "double-line" notation** to indicate vector quantities; like this:



It's almost always possible to add an extra line to a letter; this extra line denotes a vector quantity. I feel this is "tidier" and easier to write than drawing arrows over or under a letter. *Please use this notation in your homework and notes; I will use it on the chalkboard.*

In typeset documents (like this one) I will use boldface ($\mathbf{v}$) to denote a vector, and italic (v) to denote a scalar. This is standard mathematical typesetting practice.

It is very important to denote **vector** quantities as **vectors**!
I want you to always use vector notation where appropriate!

2.1.1 Subscripts

We will typically use a *right subscript* to refer to a point or body. For example, $\mathbf{r}_A$ denotes the position of point A, while $\boldsymbol{\omega}_3$ refers to the angular velocity of link 3, and $\mathbf{a}_A$ denotes the acceleration of point A.

A position vector from point A to point B will be written as $\mathbf{r}_{BA}$. Think of it as "vector to point B from A."

2.1.2 Superscripts

Where needed, we will use a *left superscript* to denote the *relativity* of a quantity. For example, to specify the velocity of point B relative to body 2, one would write ${}^2\mathbf{v}_B$. Note that a body and a reference frame are the same thing (see below).

2.2 Reference Frames & Bodies

A *reference frame* and a *body* are kinematically the same thing. Period.

Consider spoked wheel 2 rolling on fixed surface 1 as shown in Figure 1. Bead B slides out along one spoke at speed v as shown.

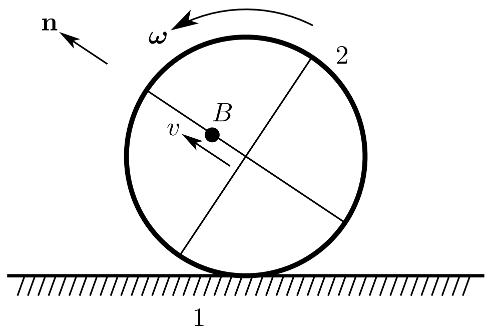


Figure 1: Spoked wheel rolling on plane.

To denote the velocity of B relative to wheel 2, we would write

$$ {}^2\mathbf{v}_B = v\mathbf{n}, \tag{1}$$

where $\mathbf{n}$ is a unit vector aligned with the spoke. Note that the velocity of B relative to body 1 would be quite different, and would involve the rolling of the wheel.

2.2.1 Notation for Motion Relative to a Fixed Body

When the motion of a point or body is relative to a fixed body (*e.g.* frame of a mechanism), the superscript indicating relativity is usually omitted. For example, in the above wheel, the velocity of B relative to body 1 (the fixed surface) would be expressed simply $\mathbf{v}_B$.

2.3 Relativity and Time Derivatives

The value of a time derivative depends on the reference frame (body) in which the derivative is computed. For example, point P may be fixed in body B, so ${}^B\mathbf{v}_P = 0$. But if body B is itself moving relative to body A, then ${}^A\mathbf{v}_P \neq 0$. That is, since velocity of P is the time derivative of its position, $\mathbf{r}_P$, then

$$\frac{{}^A d\mathbf{r}_P}{dt} \neq \frac{{}^B d\mathbf{r}_P}{dt}$$

Reference frame (body) is important.

3 Velocity and Angular Velocity

You all have an understanding of *velocity*; it is the time derivative of position. Note that velocity is a **vector**—we will typically express vectors as components along unit vectors.

Generalized *angular velocity* is not as clear. Fortunately in ME 314 we will always be dealing with *simple angular velocity* which—as its name implies—much simpler.

Finally, two important "restrictions"...

- A **POINT** can only have a **VELOCITY**, never an ANGULAR VELOCITY
- A **BODY** (or frame or link) can only have an **ANGULAR VELOCITY**, never a VELOCITY

3.1 Definition of Angular Velocity

If $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$ is a right-handed (dextral) set of mutually perpendicular unit vectors fixed in body B, the *angular velocity* ${}^R\boldsymbol{\omega}_B$ in a reference frame R may be defined as

$$ {}^R\boldsymbol{\omega}_B = \dot{\mathbf{b}}_2 \cdot \mathbf{b_3}\mathbf{b_1} + \dot{\mathbf{b}}_3 \cdot \mathbf{b_1}\mathbf{b_2} + \dot{\mathbf{b}}_1 \cdot \mathbf{b_2}\mathbf{b_3}, \tag{2} $$

where $\dot{\mathbf{b}}_i$ denotes the derivative of $\mathbf{b}_i$ with respect to time t in reference frame R. By means of a proof which is not shown here, it is true that, if $\mathbf{c}$ is any vector fixed in B,

$$ \boxed{\frac{{}^R d\mathbf{c}}{dt} = {}^R\boldsymbol{\omega}_B \times \mathbf{c}} \tag{3} $$

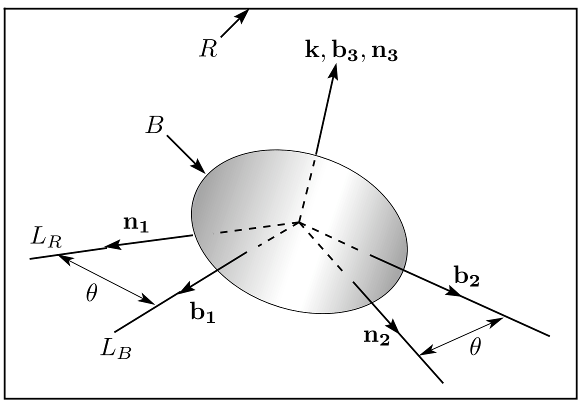


Figure 2: Simple angular velocity.

3.2 Simple Angular Velocity, Angular Speed

When a rigid body B moves in a reference frame R in such a way that there exists a unit vector $\mathbf{k}$ whose orientation in both B and R is independent of t, B is said to have a *simple angular velocity* in R, and this angular velocity can be expressed as (see Figure 2)

$$ {}^R\boldsymbol{\omega}_B = {}^R\omega_B \mathbf{k} \tag{4} $$

where

$$ {}^R\omega_B = \frac{d\theta}{dt}, \tag{5} $$

and θ is the radian measure of the angle between a line L_R whose orientation is fixed in R and a line L_B similarly fixed in B, both lines being perpendicular to $\mathbf{k}$ (see Figure 2) and θ being regarded as positive when the angle can be generated by a rotation of B during which a right-handed screw parallel to $\mathbf{k}$ and rigidly attached to B advances in the direction of $\mathbf{k}$. ${}^R\omega_B$ is called the *angular speed* of B in R for $\mathbf{k}$.

3.3 Derivatives in Two Reference Frames

If R and S are any references frames (bodies), the derivatives of a vector $\mathbf{v}$ with respect to t in R and S are related to each other as follows:

$$\frac{{}^R d\mathbf{v}}{dt} = \frac{{}^S d\mathbf{v}}{dt} + {}^R\boldsymbol{\omega}_S \times \mathbf{v} \tag{6}$$

where ${}^R\boldsymbol{\omega}_S$ is the angular velocity of S in R.

This equation will lead to the extremely useful "Two Points on a Rigid Body" equation in the next Section.

3.4 Addition Theorem

The *addition theorem for angular velocities* states that, if A, B, and C are three bodies (reference frames), then the angular velocity of body C relative to body A can be expressed as

$${}^A\boldsymbol{\omega}_C = {}^A\boldsymbol{\omega}_B + {}^B\boldsymbol{\omega}_C. \tag{7}$$

That is, angular velocities simply add. Of course, they must all be expressed using the same unit vector system. For planar systems with simple angular velocity this is usually not a consideration.

This concludes the "preliminaries." In the next section are the techniques we shall use to analyze the mechanisms.

4 Kinematical Theorems

This is the heart of the material. The two basic kinematical equations to follow will be used to analyze the problems in this course. This is a far simpler approach than that of the textbook.

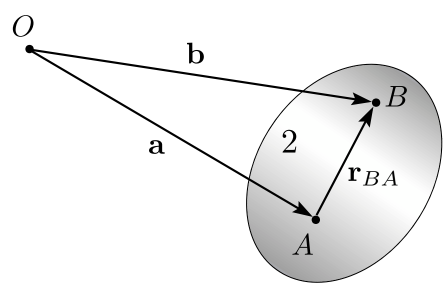


Figure 3: Points A and B on body 2.

4.1 Two Points on a Body

If A and B are points *fixed* on body 2, the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of A and B are related to each other as follows

$$\boxed{\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_2 \times \mathbf{r}_{BA}} \tag{8}$$

where $\boldsymbol{\omega}_2$ is the angular velocity of body 2, and $\mathbf{r}_{BA}$ is the position vector from A to B).

Proof: Let O be a fixed point (see Figure 3), $\mathbf{a}$ the position vector of A relative to O, and $\mathbf{b}$ the position vector of B relative to O. Then, by definition,

$$\mathbf{v}_A = \frac{d\mathbf{a}}{dt} \tag{9}$$

and

$$\mathbf{v}_B = \frac{d\mathbf{b}}{dt}. \tag{10}$$

Now (see Figure 3)

$$\mathbf{a} = \mathbf{b} + \mathbf{r}_{AB}, \tag{11}$$

hence

$$\mathbf{v}_A = \frac{d\,\mathbf{b}}{dt} + \frac{d\,\mathbf{r}_{BA}}{dt} = \mathbf{v}_B + \frac{d\,\mathbf{r}_{BA}}{dt}. \tag{12}$$

Since $\mathbf{r}_{BA}$ is a vector fixed in 2, using equation (3) we have

$$\frac{d\,\mathbf{r}_{BA}}{dt} = \boldsymbol{\omega}_2 \times \mathbf{r}_{BA}. \tag{13}$$

Substitute from (13) into (12). Equation (8) is useful in virtually *ALL* mechanism analysis situations.

4.2 Example—Two Points on a Body

Consider the four-bar linkage shown in Figure 4. This is the same mechanism of text Example 3.1 on p. 107, where it is solved graphically. The analytical solution will be presented here.

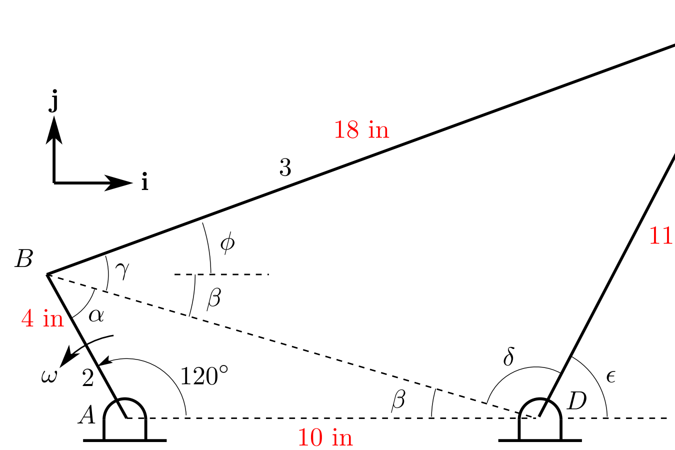


Figure 4: Four-bar linkage.

4.2.1 Geometry.

The angle of link two is 120°, and the link lengths are given in red. We are going to need to form position vectors between A, B, C, and D, so we will need angles ϵ and ϕ.

To find ϕ and ϵ we must first find other angles which are indicated on Figure 5. This is done using the laws of cosines and sines; the results are

$$\begin{aligned}
\alpha &= 43.90^\circ \\
\beta &= 16.10^\circ \\
\gamma &= 37.02^\circ \\
\delta &= 99.85^\circ \\
\epsilon &= 64.05^\circ \\
\phi &= 20.92^\circ
\end{aligned}$$

I will typically give you the geometrical information (angles & lengths) when I assign a problem.

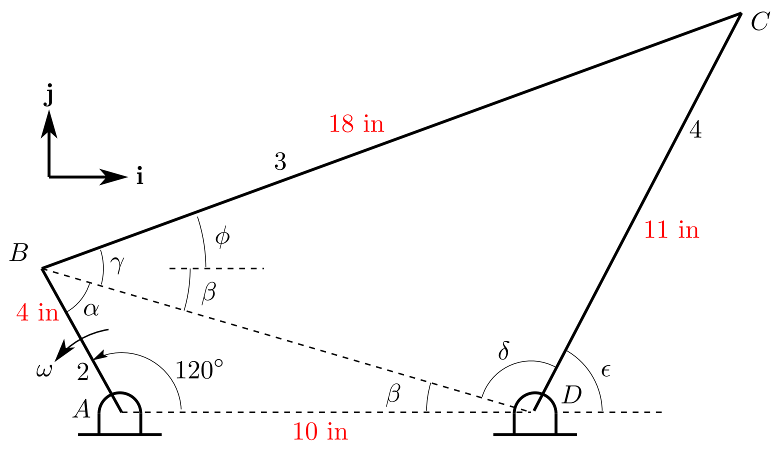


Figure 5: Four-bar linkage.

4.2.2 Method of Solution

The four-bar linkage is shown again above for convenience. We are told that link 2 rotates at 900 rpm CCW, hence

$$\boldsymbol{\omega}_2 = \left[(900 \text{ rpm}) \times \frac{2\pi}{60}\right] \mathbf{k} = 94.25\mathbf{k} \text{ rad/s} \tag{14}$$

where the sense of $\boldsymbol{\omega}_2$ is determined by the right-hand rule. We wish to find the velocities of B, C, and angular velocities of links 3 and 4.

Procedure. The solution will proceed as follows: we first find the velocity of B, then write two expressions for the velocity of C—one from B and one from D—then equate them to solve the problem.

Specific steps:

1. Since $\mathbf{v}_A$, $\boldsymbol{\omega}_2$, and $\mathbf{r}_{BA}$ are known, find $\mathbf{v}_B$ using (8)
2. Since $\mathbf{v}_B$ is now known, $\mathbf{r}_{CB}$ is known, and the *DIRECTION* of $\boldsymbol{\omega}_3$ is known (it's in the $\mathbf{k}$ direction), use (8) to write an expression for $\mathbf{v}_C$ that will contain *ONE* unknown (scalar ω_3)
3. Since $\mathbf{v}_D$ and $\mathbf{r}_{CD}$ are known, and the *DIRECTION* of $\boldsymbol{\omega}_4$ is known (it's also in the $\mathbf{k}$ direction), use (8) to write *another* expression for $\mathbf{v}_C$ that will contain a *SECOND* unknown (scalar ω_4)
4. Equate both expressions for $\mathbf{v}_C$ and solve the two equations ($\mathbf{i}$ and $\mathbf{j}$ equations) simultaneously for ω_3 and ω_4
5. Use ω_3 or ω_4 to find $\mathbf{v}_C$

Solution:

Since $\mathbf{r}_{BA} = -2\,\mathbf{i} + 3.46\,\mathbf{j}$ in, we find the velocity of B as

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_2 \times \mathbf{r}_{BA} = -326.48\,\mathbf{i} - 188.50\,\mathbf{j} \text{ in/s} \tag{15}$$

Position vector $\mathbf{r}_{CB} = 16.81\,\mathbf{i} + 6.43\,\mathbf{j}$ in, and angular velocity $\boldsymbol{\omega}_3 = \omega_3\,\mathbf{k}$, so

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_3 \times \mathbf{r}_{CB} = -326.48\,\mathbf{i} - 188.50\,\mathbf{j} - 6.43\omega_3\,\mathbf{i} + 16.81\omega_3\,\mathbf{j} \text{ in/s} \tag{16}$$

The other equation for $\mathbf{v}_C$ starts from point D:

$$\mathbf{v}_C = \mathbf{v}_D + \boldsymbol{\omega}_4 \times \mathbf{r}_{CD} = \omega_4\,\mathbf{k} \times 4.81\,\mathbf{i} + 9.89\,\mathbf{j} = -9.89\omega_4\,\mathbf{i} + 4.81\omega_4\,\mathbf{j} \text{ in/s} \tag{17}$$

Equating (16) and (17) yields

$$-326.48\,\mathbf{i} - 188.50\,\mathbf{j} - 6.43\omega_3\,\mathbf{i} + 16.81\omega_3\,\mathbf{j} = -9.89\omega_4\,\mathbf{i} + 4.81\omega_4\,\mathbf{j} \tag{18}$$

Equation (18) is really *TWO* equations ($\mathbf{i}$ and $\mathbf{j}$) and contains *TWO* unknowns (ω_3 and ω_4) so it can be solved.

I use MATLAB to solve these, and I would express (18) as

$$\mathbf{i}: \; -6.43\omega_3 + 9.89\omega_4 = 326.48 \tag{19}$$

$$\mathbf{j}: \; 16.81\omega_3 - 4.81\omega_4 = 188.50 \tag{20}$$

which can be put into matrix form

$$\begin{bmatrix} -6.43 & 9.89 \\ 16.81 & -4.81 \end{bmatrix} \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} 326.48 \\ 188.50 \end{bmatrix} \tag{21}$$

This is the classic linear algebra problem $\mathbf{Ax} = \mathbf{b}$, so $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. My results were

$$\boxed{\begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} 25.38 \\ 49.50 \end{bmatrix} \text{ rad/s}} \tag{22}$$

Those values agree with the text, but mine are more accurate.

To find the velocity of C, use either (16) or (17), although (17) is simpler. So we have

$$\boxed{\mathbf{v}_C = -9.89\omega_4\,\mathbf{i} + 4.81\omega_4\,\mathbf{j} = -489.62\,\mathbf{i} + 238.28\,\mathbf{j} \text{ in/s} = -40.80\,\mathbf{i} + 19.86\,\mathbf{j} \text{ ft/s}} \tag{23}$$

Common sense. The angular velocities of (22) are both positive (CCW by RH rule), and looking at the mechanism this makes sense (although ω_3 is hard to see). Also expression (23) shows that point C is moving mostly left and somewhat up. This also makes sense.

Other points. In text Example 3.1 links 3 and 4 are actually plates with additional points E and F. We can easily find the velocity of E and F since we already know the velocities of another point on those links (B and D, for example), and we know angular velocities $\boldsymbol{\omega}_3$ and $\boldsymbol{\omega}_4$.

4.3 What Mechanisms Can be Solved using Two Points on a Body?

The following mechanisms CAN be completely solved using only equation (8):

- Those with only pin joints
- Those with only pin joints and sliding relative to a *fixed* body
- Rolling contact (no slip): the velocity of the contact points on both bodies is the same

The following mechanisms CANNOT be completely solved using only equation (8):

- Those with sliding motion relative to a *rotating* body

To solve the latter problems another "kinematical theorem" is needed: **"One Point Moving on a Body."**

Before presenting that principle, consider another mechanism—very similar to the previous one but with *SLIDING RELATIVE TO A FIXED BODY.*

4.4 Example—Mechanism with Sliding Contact Relative to Fixed Surface

Consider Figure 6, which is the same as Figure 5 except that link 4 has been replaced with a fixed circular surface of radius 11 inches centered at D.

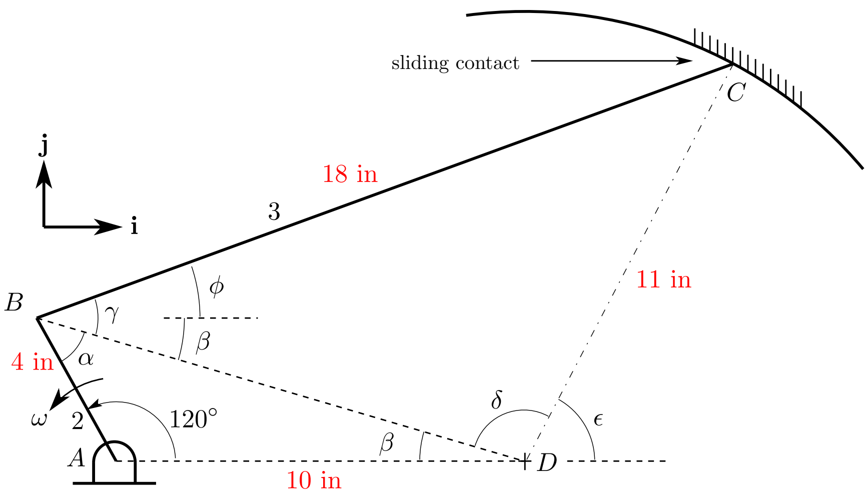


Figure 6: Four-bar linkage with sliding relative to a *FIXED* surface at C.

By inspection, the motion of point C in Figure 6 is exactly the same as it is in Figure 4. Indeed, analysis of this new mechanism is *almost* the same, but there is a difference in the specification of $\mathbf{v}_C$, the velocity of C.

The "two points on a body" equations for $\mathbf{v}_B$ and $\mathbf{v}_C$ are exactly the same as before in (15) and (16). However, since there is really no $\boldsymbol{\omega}_4$ equation (17) must be stated differently.

Since the circular surface has center of curvature at D, hence the velocity of C must be perpendicular to line CD (which is at angle $\epsilon = 64.05°$). When the **DIRECTION** of a vector is known (but not its **MAGNITUDE**) one should write it using a symbolic (unknown) magnitude in a known (numeric) direction.

Examining Figure 6, the motion of C will be to the left (and up), hence it probably makes sense to use this as the direction. The angle of this direction will be $64.05° + 90° = 154.05°$, so we have

$$\mathbf{v}_C = v_C\,(\cos 154.05°\,\mathbf{i} + \sin 154.05°\,\mathbf{j}) = v_C\,(-0.899\,\mathbf{i} + 0.438\,\mathbf{j}) = -0.899 v_C\,\mathbf{i} + 0.438 v_C\,\mathbf{j} \tag{24}$$

Now equations (15), (16), and (24) will be used to solve the problem. Instead of solving for ω_3 and ω_4, we now solve for ω_3 and v_C.

I encourage you to solve this problem, and verify that $\mathbf{v}_C$ is the same as for Figure 5.

4.5 The Second Kinematic Principle: One Point Moving on a Body

This principle is needed when there is a point sliding on a body which itself rotates.

If a point P moves relative to rigid body 3 while body 3 itself moves (*i.e.* rotates and translates), the velocity $\mathbf{v}_P$ of P (relative to ground) is related to the velocity ${}^3\mathbf{v}_P$ of P relative to 3 as follows:

$$\boxed{\mathbf{v}_P = \mathbf{v}_{P_3} + {}^3\mathbf{v}_P} \tag{25}$$

where $\mathbf{v}_{P_3}$ denotes the velocity of point P but fixed on body 3, and ${}^3\mathbf{v}_P$ denotes the velocity of P relative to body 3.

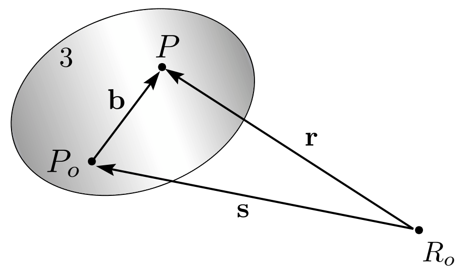


Figure 7: Point P moving on body 3.

Proof: Let R_o be a fixed point, B_o be a point fixed in body 3, $\mathbf{b}$ the position vector of B relative to B_o, $\mathbf{r}$ the position vector of B relative to R_o, and $\mathbf{s}$ the position vector of B_o relative to R_o, as shown in Figure 7. Then the velocities $\mathbf{v}_B$, ${}^3\mathbf{v}_B$, and $\mathbf{v}_{B_o}$, each expressed as a derivative of a position, are given by

$$\mathbf{v}_P = \frac{d\mathbf{r}}{dt} \tag{26}$$

$${}^3\mathbf{v}_P = \frac{{}^3d\mathbf{b}}{dt} \tag{27}$$

$$\mathbf{v}_{P_o} = \frac{d\mathbf{s}}{dt} \tag{28}$$

Now (see Figure 7)

$$\mathbf{r} = \mathbf{s} + \mathbf{b}, \tag{29}$$

so consequently

$$\mathbf{v}_P = \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{b}}{dt} \tag{30}$$

$$= \mathbf{v}_{P_o} + \frac{{}^3d\mathbf{b}}{dt} \underbrace{+}_{(3)} \boldsymbol{\omega}_3 \times \mathbf{b} \tag{31}$$

$$= \mathbf{v}_{P_o} + {}^3\mathbf{v}_P + \boldsymbol{\omega}_3 \times \mathbf{b} \tag{32}$$

Point P_o may always be taken as P_3 —that is, as the point of 3 with which P coincides. Then $\mathbf{b} = 0$ and

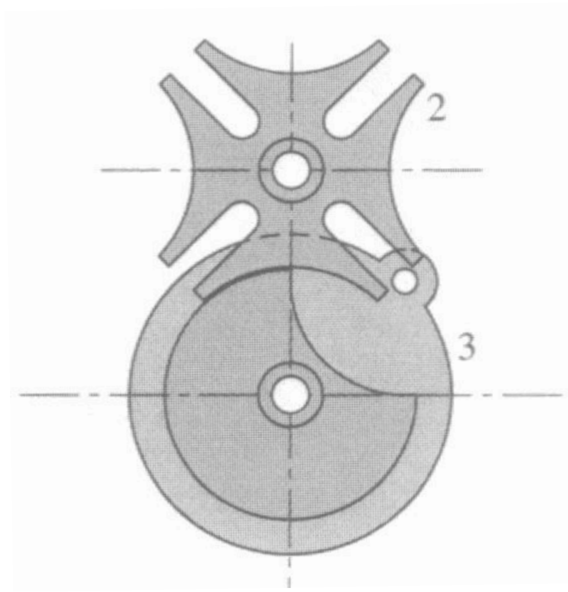
$$\boxed{\mathbf{v}_P = \mathbf{v}_{P_3} + {}^3\mathbf{v}_P} \tag{33}$$

In applying this principle, there will be two bodies that are sliding relative to each other. You must identify a point on one body **WHOSE PATH IS KNOWN** relative to the other body. This will be the point "P" and body "3" used in equation (33).

This requires **inversion**: fix one body and examine the motion of a point. Sometimes it is obvious, other times not so obvious.

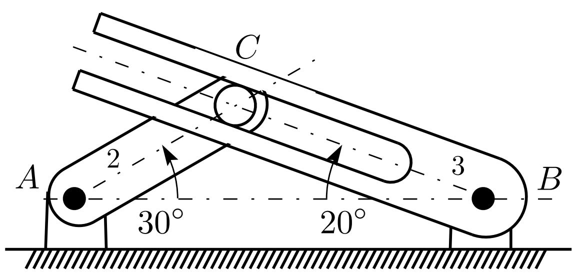
4.6 Example—One Point Moving on a Body

Consider the Geneva Stop mechanism (text p. 22) is shown below:



Link 3 is typically driven at constant speed, and link 2 will thereby index at 90° intervals. This is due to the pin sliding along the rotating slot.

A portion of the Geneva Stop is shown in the simplified mechanism below:



The dimensions at this instant are:

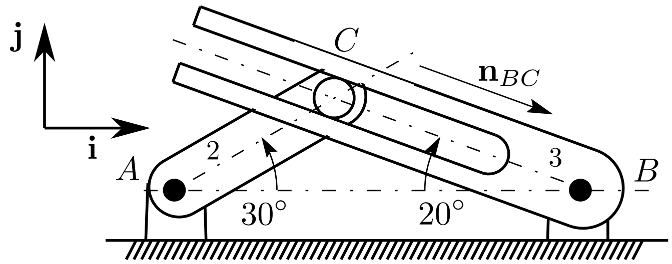
$$AB = 2 \text{ in}$$
$$AC = 0.893 \text{ in}$$
$$BC = 1.305 \text{ in}$$

The applied motion is:

$$\omega_2 = 955 \text{ rpm (100 rad/sec) CCW}$$

so this mechanism is zipping along pretty quickly...

We want to find the angular velocity $\boldsymbol{\omega}_3$ and the speed of the pin in the slot at this instant.



Solution.

1. Points A and C are both fixed on link 2; hence their velocity can be related using the "2 pts on a body" equation, and we have:

$$\mathbf{v}_C = \underbrace{\mathbf{v}_A}_{=0} + \boldsymbol{\omega}_2 \times \mathbf{r}_{CA} \tag{34}$$

$$= \boldsymbol{\omega}_2 \times \mathbf{r}_{CA} \tag{35}$$

where

$$\boldsymbol{\omega}_2 = 100\mathbf{k} \text{ rad/s} \tag{36}$$

$$\mathbf{r}_{CA} = r_{CA}(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) \text{ in} \tag{37}$$

2. However, points B and C are *NOT* fixed on link 3; we *CANNOT* relate their velocities using the "2 pts on a body" equation. This is case where there is *sliding contact with a rotating body.* We must use the "1 pt moving on a body" equation.

Critical Next Step: The two bodies that are sliding here are bodies 2 and 3. We must identify **the point** on one of those bodies **whose path we know** relative to the other body. See the "box" at the bottom of p. 9.

Look at point C and body 3; invert the mechanism by fixing link 3 and see what the motion of C can be. The answer here is simply "along the slot," in the direction of BC. So you know the path of C relative to 3.

So...the "one point moving on a body" equation will be

$$\mathbf{v}_C = \mathbf{v}_{C_3} + {}^3\mathbf{v}_C \tag{38}$$

where

- $\mathbf{v}_{C_3}$= velocity of point on body 3 coincident with C at this instant
- ${}^3\mathbf{v}_C$= velocity of point C relative to body 3

Points B and C_3 are BOTH fixed to link 3, and $\boldsymbol{\omega}_3 = \omega_3\,\mathbf{k}$, so we can relate them using the "two points on a body" equation:

$$\mathbf{v}_{C_3} = \underbrace{\mathbf{v}_B}_{=0} + \boldsymbol{\omega}_3 \times \mathbf{r}_{CB} \tag{39}$$

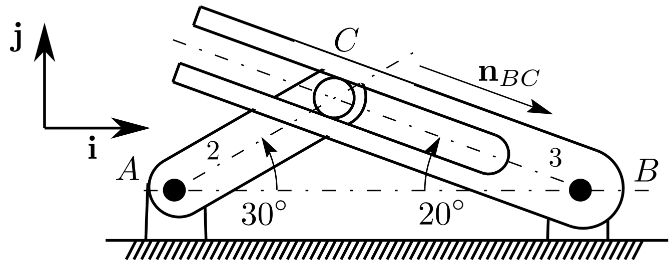
$$= \boldsymbol{\omega}_3 \times \mathbf{r}_{CB} = \omega_3\mathbf{k} \times 1.305(-\cos 20^\circ \mathbf{i} + \sin 20^\circ \mathbf{j}) \tag{40}$$

Also, the direction of relative velocity ${}^3\mathbf{v}_C$ is known: it's along slot BC. You can therefore write this velocity as an unknown magnitude and a known direction. I'll first do this using a unit vector defined as $\mathbf{n}_{BC}$ to denote the direction from C to B. Then the relative velocity is

$${}^3\mathbf{v}_C = {}^3v_C\ \mathbf{n}_{BC}, \tag{41}$$

and unit vector $\mathbf{n}_{BC}$ can be expressed in terms of $\mathbf{i}$ and $\mathbf{j}$ as

$$\mathbf{n}_{BC} = \cos 20^\circ \mathbf{i} - \sin 20^\circ \mathbf{j} \tag{42}$$



Finally, there are two unknowns in (38), since velocity $\mathbf{v}_C$ is known from (35). Evaluating all expressions numerically, we have first from (35),

$$\mathbf{v}_C = -44.65\mathbf{i} + 77.3361\mathbf{j} \text{ in/s.} \tag{43}$$

Then from (40),

$$\mathbf{v}_{C_3} = -0.4463\,\omega_3\mathbf{i} - 1.2263\,\omega_3\mathbf{j} \text{ in/s} \tag{44}$$

Next, from (41) and (42),

$$^3\mathbf{v}_C = 0.9397\,{}^3v_C\mathbf{i} - 0.3420\,{}^3v_C\mathbf{j} \text{ in/s} \tag{45}$$

Using (38) we get the second equation for the velocity of C, and substituting (44) and (45) in yields:

$$\mathbf{v}_C = (-0.4463\,\omega_3 + 0.9397\,{}^3v_C)\mathbf{i} + (-1.2263\,\omega_3 - 0.3420\,{}^3v_C)\mathbf{j} \text{ in/s} \tag{46}$$

Finally, equate (46) and (43) — since they're both equations for $\mathbf{v}_C$ — and you have

$$(-0.4463\,\omega_3 + 0.9397\,{}^3v_C)\mathbf{i} + (-1.2263\,\omega_3 - 0.3420\,{}^3v_C)\mathbf{j} = -44.65\mathbf{i} + 77.3361\mathbf{j} \text{ in/s} \tag{47}$$

Separating into $\mathbf{i}$ and $\mathbf{j}$ equations,

$$\mathbf{i} : -0.4463\,\omega_3 + 0.9397\,{}^3v_C = -44.65 \tag{48}$$
$$\mathbf{j} : -1.2263\,\omega_3 - 0.3420\,{}^3v_C = 77.3361 \tag{49}$$

Expressing these in matrix form (not required, but I usually solve them this way),

$$\begin{bmatrix} -0.4463 & 0.9397 \\ -1.2263 & -0.3420 \end{bmatrix} \begin{bmatrix} \omega_3 \\ {}^3v_C \end{bmatrix} = \begin{bmatrix} -44.65 \\ 77.3361 \end{bmatrix} \tag{50}$$

Solution using MATLAB is

$$\begin{bmatrix} \omega_3 \\ {}^3v_C \end{bmatrix} = \begin{bmatrix} -43.9869 \text{ rad/s} \\ -68.4063 \text{ in/s} \end{bmatrix} \tag{51}$$

Going back to our original convention for $\boldsymbol{\omega}_3$ and $^3\mathbf{v}_C$, the final vector result is

$$\boldsymbol{\omega}_3 = \omega_3\mathbf{k} = -43.9869\,\mathbf{k} \text{ rad/s} \tag{52}$$
$$^3\mathbf{v}_C = 0.9397\,{}^3v_C\mathbf{i} - 0.3420\,{}^3v_C\mathbf{j} \tag{53}$$
$$= -64.2814\mathbf{i} + 23.3950\mathbf{j} \text{ in/s} \tag{54}$$

The result for ^3v_C indicates the pin is sliding relative to the slot "leftwards" ($-\mathbf{n}_{BC}$ direction) at 68 in/s, and link 3 is rotating CW at 44 rad/s (420 rpm). These make sense to me.

4.7 Again—When to Use Which Equation?

Again, here are a few guidelines on which equation to use, and how to use them.

4.7.1 Two Points on a Body

Use the "2 pts. on a body" equation when

- A mechanism has only pin joints, or sliding on a *FIXED* surface (four-bar with pin joints, or a slider crank)
- A portion of a mechanism with only pin joints (points O_2 and A of Problem 3.15)
- Rolling contact (no slip): the velocity of the contact points on both bodies is the same

4.7.2 One Point Moving on a Body

Use the "one point moving on a body" equation when

- A mechanism involves sliding motion relative to a *ROTATING* body (Problem 3.15 or the "Geneva Stop" just considered)

5 Acceleration

The determination of acceleration is important because of the relationship to forces. Acceleration is the time derivative of velocity,

$$ {}^{R}\mathbf{a}_P = \frac{{}^{R}d\mathbf{v}_P}{dt} \tag{55}$$

However, just as for velocity, the determination of acceleration in a mechanism is often facilitated by the use of the following two kinematical relationships.

5.1 Two Points of a Rigid Body

If P and Q are points fixed on rigid body B, the accelerations $\mathbf{a}_P$ and $\mathbf{a}_Q$ of P and Q in a reference frame R are related to each other by

$$\mathbf{a}_P = \mathbf{a}_Q + \underbrace{\boldsymbol{\alpha} \times \mathbf{r}}_{\text{tangential}} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}_{\text{normal}} \tag{56}$$

where (as before) $\mathbf{r}$ is the position vector from Q to P, $\boldsymbol{\omega}$ is the angular velocity of body B in R, and $\boldsymbol{\alpha}$ is the angular acceleration of body B in R. The terms labeled "tangential" and "normal" correspond to the tangential acceleration and normal acceleration, respectively. Remember that if the path is curved, you *ALWAYS* have normal acceleration!

5.1.1 Angular Acceleration

The *angular acceleration* ${}^{R}\boldsymbol{\alpha}_B$ of a rigid body B in a reference frame R is define as the time derivative in R of the angular velocity of B in R:

$$ {}^{R}\boldsymbol{\alpha}_B = \frac{{}^{R}d\, {}^{R}\boldsymbol{\omega}_B}{dt} \tag{57}$$

When ${}^{R}\boldsymbol{\omega}_B = {}^{R}\omega_B\mathbf{k} = \dot{\theta}\mathbf{k}$ is *simple angular velocity* (unit vector $\mathbf{k}$ is fixed; see Section 3.2), the angular acceleration is just

$$ {}^{R}\boldsymbol{\alpha}_B = \ddot{\theta}\mathbf{k} \tag{58}$$

This is typically the case with the planar mechanisms of ME 314.

5.1.2 Proof of Two Points of a Rigid Body

Coming sometime...maybe

5.2 Example - Two Points of a Rigid Body

Consider text Problem 4.7, as shown below.

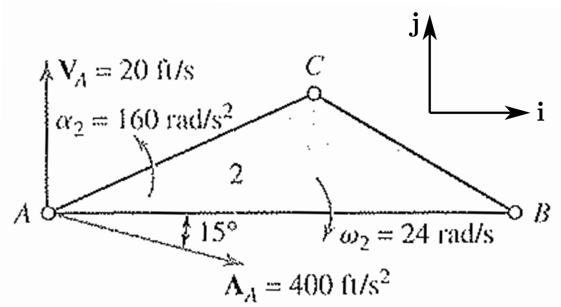


Figure P4.7 $R_{BA} = 16$ in, $R_{CA} = 10$ in, $R_{CB} = 8$ in.

The velocity and acceleration of point A is given, and the angular velocity ω_2 and angular acceleration α_2 of body 2 are given. Points A, B, and C are all fixed to body 2. Let's find the acceleration $\mathbf{a}_B$ of point B.

Solution. From the "Two Points..." equation, we have that

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_2 \times \mathbf{r}_{BA} + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{r}_{BA}) \tag{59}$$

where

$$\mathbf{a}_A = 400(\cos 15^\circ\,\mathbf{i} - \sin 15^\circ\,\mathbf{j}) \text{ ft/s}^2 \tag{60}$$

$$\boldsymbol{\alpha}_2 = 160\,\mathbf{k} \text{ rad/s}^2 \tag{61}$$

$$\boldsymbol{\omega}_2 = -24\,\mathbf{k} \text{ rad/s} \tag{62}$$

$$\mathbf{r}_{BA} = 16\,\mathbf{i} \text{ in} \tag{63}$$

Tangential acceleration: The tangential acceleration is

$$\boldsymbol{\alpha}_2 \times \mathbf{r}_{BA} = 160\,\mathbf{k} \times\ 16\,\mathbf{i} = 2,560\,\mathbf{j} \text{ in/s}^2 = 213\,\mathbf{j} \text{ ft/s}^2 \tag{64}$$

Note that the tangential acceleration has magnitude $\alpha_2 r_{BA}$ and is perpendicular to AB.

Normal acceleration: The normal acceleration is

$$\boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{r}_{BA}) = -24\,\mathbf{k} \times (-24\,\mathbf{k} \times 16\,\mathbf{i}) = -9,216\,\mathbf{i} \text{ in/s}^2 = -768\,\mathbf{i} \text{ ft/s}^2 \tag{65}$$

Final Result: Adding everything up (make sure the units are compatible), we have

$$\boxed{\mathbf{a}_B = -382\,\mathbf{i} + 109\,\mathbf{j} \text{ ft/s}^2} \tag{66}$$

5.3 One Point Moving on a Rigid Body

If a point P moves on a rigid body 3 while in turn body 3 moves (translates and rotates) relative to ground, the acceleration $\mathbf{a}_P$ of P is given by

$$\boxed{\mathbf{a}_P = \mathbf{a}_{P_3} + {}^3\mathbf{a}_P + \underbrace{2\boldsymbol{\omega}_3 \times {}^3\mathbf{v}_P}_{\text{Coriolis Accel.}}} \tag{67}$$

where

- $\mathbf{a}_{P_3}$ is the the acceleration of point P but *attached to body 3*
- ${}^3\mathbf{a}_P$ is the acceleration of P relative to body 3 (the "relative acceleration")
- $\boldsymbol{\omega}_3$ is the angular velocity of 3
- ${}^3\mathbf{v}_P$ is the velocity of P relative to 3 (already known from velocity analysis)

As indicated in (67), the term $(2\boldsymbol{\omega}_3 \times {}^3\mathbf{v}_P)$ is referred to as the "Coriolis acceleration," and occurs because of the rotation of the relative velocity vector.

5.3.1 Relative Acceleration ${}^3\mathbf{a}_P$

The "relative acceleration" term in (67) deserves a little more discussion. The key consideration here is that **WE WILL ALWAYS KNOW THE PATH OF POINT P RELATIVE TO BODY 3!!** This path will be either **STRAIGHT** or **CURVED**.

(a) Straight Relative Path. This is the easiest—the relative acceleration is simply directed along the path, and one can write an equation like

$$ {}^3\mathbf{a}_P = {}^3a_P\mathbf{t}, \tag{68}$$

where

- $\mathbf{t}$ is a unit vector tangential to the path in the direction of the relative motion
- ${}^3a_P$ is the magnitude of the relative acceleration

(b) Curved Relative Path. If point P describes a curved relative path on body 3, there will be both *NORMAL* and *TANGENTIAL* relative acceleration, and one must write an equation like this

$$ {}^3\mathbf{a}_P = \left({}^3a_P\right)_t\mathbf{t} + \frac{\left({}^3v_P\right)^2}{\rho}\mathbf{n}, \tag{69}$$

where

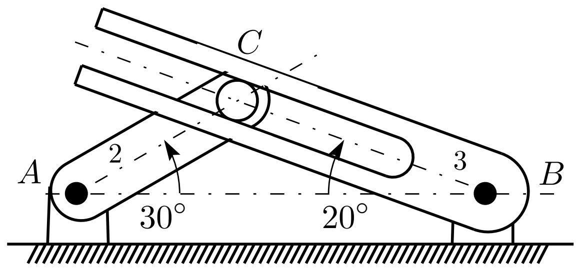
- $\mathbf{t}$ is a unit vector tangential to the path in the direction of the relative motion
- $\left({}^3a_P\right)_t$ is the magnitude of the tangential acceleration
- ρ is the *RADIUS of CURVATURE* of the relative path (*e.g.* radius of circle for circular relative path)
- $\mathbf{n}$ is a unit vector directed *TOWARD* the center of curvature
- ${}^3v_P$ is the relative speed of point P (from the velocity analysis)

5.4 When to Use the "2 Points..." or "1 Point Moving..." Equation?

Same guidelines as in Section 4.3.

5.5 Example of One Point Moving on a Rigid Body

Consider the same "Geneva Stop" mechanism as in Section 4.6. Link 2 is rotating at a constant $\boldsymbol{\omega}_2 = 100\mathbf{k}$ rad/s.



We wish to find angular acceleration $\boldsymbol{\alpha}_3$ and ${}^3\mathbf{a}_C$, the acceleration of the pin relative to the slot.
Since there is sliding contact relative to a rotating body, we must use the "One Point Moving on a Body" principle.
The pattern of the solution will be:

1. Realize that we know the PATH of point C relative to body 3 (along the slot)
2. The "One Point Moving..." equation will be: $\mathbf{a}_C = \mathbf{a}_{C_3} + {}^3\mathbf{a}_C + 2\boldsymbol{\omega} \times {}^3\mathbf{v}_C$
3. Find $\mathbf{a}_C$ from A using $\boldsymbol{\omega}_2$ and the "Two Points on a Body" equation
4. Express $\mathbf{a}_{C_3}$ from point B using $\boldsymbol{\omega}_3$, $\boldsymbol{\alpha}_3$ and the "Two Points on a Body" equation
5. Since we know the relative path, the term ${}^3\mathbf{a}_C$ has a known direction (along the slot)
6. Evaluate the Coriolis term $2\boldsymbol{\omega}_3 \times {}^3\mathbf{v}_C$
7. Equate and solve...

Solution. The dimensions at this instant are:

$$\begin{aligned} AB &= 2 \text{ in} \\ AC &= 0.893 \text{ in} \\ BC &= 1.305 \text{ in} \end{aligned}$$

From the velocity analysis of Section 4.6 we found that

$$\boldsymbol{\omega}_3 = -43.9869\,\mathbf{k} \text{ rad/s} \tag{70}$$

$${}^3\mathbf{v}_C = -64.2814\mathbf{i} + 23.3950\mathbf{j} \text{ in/s} \tag{71}$$

From Step 2 we have $\mathbf{a}_C = \mathbf{a}_{C_3} + {}^3\mathbf{a}_C + 2\boldsymbol{\omega} \times {}^3\mathbf{v}_C$.
From Step 3 we have (since $\boldsymbol{\alpha}_2 = 0$),

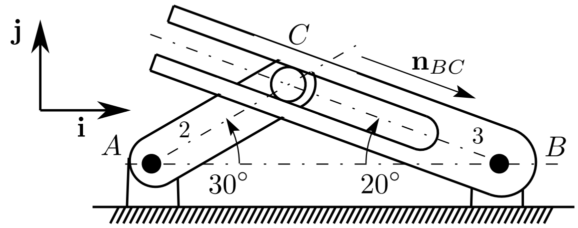
$$\mathbf{a}_C = \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{r}_{CA}) = -7734\mathbf{i} - 4465\mathbf{j} \text{ in/s}^2 \tag{72}$$

From Step 4 we have (since $\boldsymbol{\omega}_3$ is known and $\mathbf{a}_B = 0$)

$$\begin{aligned} \mathbf{a}_{C_3} &= \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{r}_{CB}) + \boldsymbol{\alpha}_3 \times \mathbf{r}_{CB} \\ &= 2373\mathbf{i} - 864\mathbf{j} - 0.4463\alpha_3\mathbf{i} - 1.2263\alpha_3\,\mathbf{j} \text{ in/s}^2 \end{aligned} \tag{73}$$

From Step 5 we have the relative acceleration along the slot (straight-line relative path),

$${}^3\mathbf{a}_C = {}^3a_C\mathbf{n}_{BC} = {}^3a_C(0.9397\mathbf{i} - 0.3420\mathbf{j}) \tag{74}$$



From Step 6 evaluate the Coriolis acceleration (comes from the rotation of the relative velocity vector),

$$2\boldsymbol{\omega}_3 \times {}^3\mathbf{v}_C = 2058\mathbf{i} + 5655\mathbf{j}\ \text{in/s}^2 \tag{75}$$

From Step 7 equate and solve, you get the following **i** and **j** equations:

$$\mathbf{i} : 0.4463\alpha_3 + 0.9397\,{}^3a_C = 12165 \tag{76}$$

$$\mathbf{j} : 1.2263\alpha_3 + 0.3420\,{}^3a_C = 9256 \tag{77}$$

In matrix form this is

$$\begin{bmatrix} 0.4463 & -0.9397 \\ 1.2263 & 0.3420 \end{bmatrix} \begin{bmatrix} \alpha_3 \\ {}^3a_C \end{bmatrix} = \begin{bmatrix} 12165 \\ 9256 \end{bmatrix} \tag{78}$$

My final result is

$$\boxed{\begin{aligned} \alpha_3 &= 9853\ \text{rad/s}^2 \quad (79) \\ {}^3a_C &= -8266\ \text{in/s}^2 \quad (80) \end{aligned}}$$

It is difficult for me to examine the mechanism and deduce if these are intuitively correct. That's one of the drawbacks of this stuff. However, I cannot find any errors, and since I made this problem up myself I don't have an "answer book."