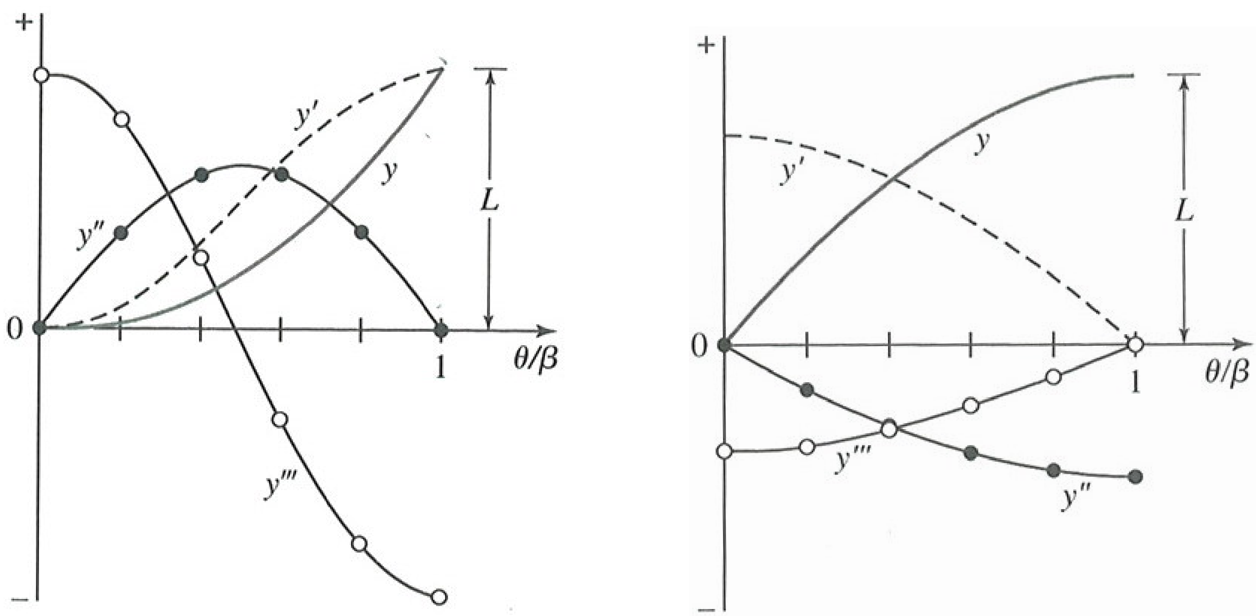
Combining Displacement Functions

1 Problem Statement

Combine a half-rise cycloidal and half-rise harmonic into a single full-rise lift of $L = 1$ inch in $\beta = 180°$ ($\beta = \pi$) of cam rotation. This is representative of motion construction and derivative matching.

2 Displacement Functions

Both displacement functions and their derivatives (kinematic coefficients) are plotted and listed in the text. The cycloidal and harmonic half-rise function we will use are plotted in text Figures 6.22(a) and 6.20(b), which are reproduced in Figure 1 below. You can visualize the two y plots fitting together (one is concave up, the other concave down).



(a) Cycloidal half-rise (Figure 6.22(a)). (b) Harmonic half-rise (Figure 6.20(b)).

Figure 1: Plots of half-rise displacement functions and their derivatives.

Consider the derivatives of each motion at the boundary between functions (end of cycloidal and start of harmonic) shown in Table 1. It appears that these functions match up well at the boundary point. For reference purposes, the

	cycloidal	harmonic
y'	nonzero	nonzero
y''	zero	zero
y'''	nonzero	nonzero

Table 1: Qualitative values of derivatives at boundary point.

displacement function for each motion is given below (y_1 is cycloidal, y_2 is harmonic):

$$y_1 = L_1 \left(\frac{\theta}{\beta_1} - \frac{1}{\pi} \sin \frac{\pi\theta}{\beta_1} \right) \tag{1}$$

$$y_2 = L_2 \sin \frac{\pi\theta}{2\beta_2} \tag{2}$$

To specify the complete motion requires the determination of **FOUR** parameters: $\begin{bmatrix} L_1 & \beta_1 & L_2 & \beta_2 \end{bmatrix}$.

3 Constraints

To solve for the **four** preceding parameters, we clearly need **four** mathematical constraints. The constraints in this problem are of two types: geometrical and derivative.

3.1 Geometrical Constraints

The lift from each motion must sum to the total desired lift of L, hence

$$\boxed{L_1 + L_2 = L} \tag{3}$$

Likewise, the angular duration of each motion must sum to the total duration of π, hence

$$\boxed{\beta_1 + \beta_2 = \pi} \tag{4}$$

3.2 Derivative Constraints

Displacement (lift) will necessarily be "matched up" between the segments, but the other derivatives may not. Examination of Table 1 shows that, while y'' is zero at the boundary (obviously equal), there are two other derivatives that are nonzero: y' and y'''.

The requirement that y' and y''' be equal at the boundary point will provide the two additional constraints needed to complete the design.

3.2.1 Derivative y'

The first derivative (first kinematic coefficient) for the cycloidal half-rise is given by text equation (6.22b) as

$$y' = \frac{L_1}{\beta_1} \left(1 - \cos \frac{\pi\theta}{\beta_1} \right) \tag{5}$$

Evaluation of (5) at $\theta/\beta = 1$ yields the "final" first derivative of segment 1:

$$y'_{1f} = \frac{2L_1}{\beta_1} \tag{6}$$

The first derivative for the harmonic half-rise is given by text equation (6.18b) as

$$y' = \frac{\pi L_2}{2\beta_2} \cos \frac{\pi\theta}{2\beta_2} \tag{7}$$

Evaluation of (7) at $\theta/\beta = 0$ yields the "initial" first derivative of segment 2:

$$y'_{2i} = \frac{\pi L_2}{2\beta_2} \tag{8}$$

Equating (6) and (8) forces the first derivatives to be equal, and results in

$$\boxed{\frac{4L_1}{\beta_1} = \frac{\pi L_2}{\beta_2}} \tag{9}$$

3.2.2 Derivative y'''

The third derivative (third kinematic coefficient) for the cycloidal half-rise is given by text equation (6.22d) as

$$y''' = \frac{\pi^2 L_1}{\beta_1^3} \cos \frac{\pi\theta}{\beta_1} \tag{10}$$

Evaluation of (10) at $\theta/\beta_1 = 1$ yields the "final" third derivative of segment 1:

$$y'''_{1f} = -\frac{\pi^2 L_1}{\beta_1^3} \tag{11}$$

The third derivative for the harmonic half-rise is given by text equation (6.19d) as

$$y''' = -\frac{\pi^3 L_2}{8\beta_2^3} \cos \frac{\pi\theta}{2\beta_2} \tag{12}$$

Evaluation of (12) at $\theta/\beta = 0$ yields the "initial" third derivative of segment 2:

$$y'''_{2i} = -\frac{\pi^3 L_2}{8\beta_2^3} \tag{13}$$

Equating (11) and (13) forces these third derivatives to agree, and results in

$$\boxed{\frac{L_1}{\beta_1^3} = \frac{\pi L_2}{8\beta_2^3}} \tag{14}$$

We now have four equations (3,4,9,14) and four unknowns $(L_1, L_2, \beta_1, \beta_2)$.

4 Solution.

While the four equations are coupled and nonlinear, they are not difficult to solve.

4.1 Durations β_1 and β_2

From (9) and (14) it can be shown that

$$2\beta_2^2 = \beta_1^2 \tag{15}$$

Eliminating β_1 using (4), we get the quadratic

$$\beta_2^2 + 2\pi\beta_2 - \pi^2 = 0 \tag{16}$$

There are two solutions to (16),

$$\beta_2 = -7.5845$$

$$\boxed{\beta_2 = 1.3013 \ \ (74.5584^\circ)} \tag{17}$$

Then β_1 can be found using (4) as

$$\boxed{\beta_1 = \pi - \beta_2 = 1.8403 \ \ (105.4416^\circ)} \tag{18}$$

4.2 Lifts L_1 and L_2

Knowing β_1 and β_2, go back to (9) and (14). You can find that

$$\boxed{L_1 = \frac{\pi\beta_1}{\pi\beta_1 + 4\beta_2} = 0.5262 \text{ in}} \tag{19}$$

And, of course,

$$\boxed{L_2 = 1 - L_1 = \frac{4\beta_2}{\pi\beta_1 + 4\beta_2} = 0.4738 \text{ in}} \tag{20}$$

I used these values in my "Cam Design MATLAB Script" to plot the motions, then to generate the y and y' necessary to plot the cam profile.