PRACTICE THE TECHNIQUES

For problems P1.1 to P1.5, investigate the use of Gaussian elimination to solve systems of two equations in two unknowns.

a. Solve the system as given.
b. Graph the given equations, and graph the transformed second equation after the elimination step.
c. Repeat Parts a and b for the system consisting of the same two equations, but given in reverse order.

P1.1 \[ \begin{align*}
4x + y &= 6, \\
-x + 2y &= 3.
\end{align*} \]

P1.2 \[ \begin{align*}
4x + y &= 9, \\
x + 2y &= 4.
\end{align*} \]

P1.3 \[ \begin{align*}
3x + y &= 4, \\
6x + 7y &= 13.
\end{align*} \]

P1.4 \[ \begin{align*}
5x + y &= 17, \\
-10x + 17y &= 4.
\end{align*} \]

P1.5 \[ \begin{align*}
3x + y &= 6, \\
x + 5y &= 16.
\end{align*} \]

For problems P1.6 to 1.10, investigate the convergence of the given fixed-point iteration formula.

a. Show the first three iterations graphically.
b. Compute the first three iterates algebraically.
c. Determine whether the conditions of the fixed-point convergence theorem are satisfied.

P1.6 \[ x = g(x) = 0.5x^3 + 0.3. \]

P1.7 \[ x = g(x) = \sin(2x). \]

P1.8 \[ x = g(x) = -0.5x^3 + 0.8. \]

P1.9 \[ x = g(x) = 1 - x^2. \]
\textbf{P1.10} \quad x = g(x) = \frac{1}{2} + \frac{1}{2} + \sin (3x)

Problems P1.11 to P1.15 illustrate the use of the Geršgorin theorem to find bounds on the eigenvalues of a matrix. Find the Geršgorin circles for each row of the given matrix. Graph the regions, and give bounds on the eigenvalues.

\begin{align*}
\textbf{P1.11} \quad A &= \begin{bmatrix}
1 & 1/8 & 1/4 \\
1/2 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix} \\
\textbf{P1.12} \quad A &= \begin{bmatrix}
1 & 3/4 & 0 \\
1/2 & 2 & -1/8 \\
0 & 1/8 & 3
\end{bmatrix} \\
\textbf{P1.13} \quad A &= \begin{bmatrix}
1 & 1/4 & 0 \\
1/4 & 2 & 1/4 \\
0 & 1/4 & 3
\end{bmatrix} \\
\textbf{P1.14} \quad A &= \begin{bmatrix}
1 & 1/3 & 1/3 \\
1/4 & 2 & 1/4 \\
1/2 & 1/4 & 3
\end{bmatrix} \\
\textbf{P1.15} \quad A &= \begin{bmatrix}
1 & -1/2 & 0 \\
1/2 & 2 & 1/8 \\
0 & 1/8 & 3
\end{bmatrix}
\end{align*}

Problems P1.16 to P1.17 illustrate the effect of round-off error in adding numbers of differing magnitudes.
\begin{enumerate}
\item Add from left to right, rounding to three digits at each step.
\item Add from right to left, rounding to three digits at each step.
\end{enumerate}

\textbf{P1.16} \quad 100 + 0.49 + 0.49
\textbf{P1.17} \quad 10.0 + 0.333 + 0.333 + 0.333

Problems P1.18 to P1.20 illustrate the effect of round-off in the quadratic formula.
\begin{enumerate}
\item Use the standard quadratic formula with rounding.
\item Use the rationalized-numerator quadratic formula with rounding.
\item Compare the results from parts a and b with the results found without rounding.
\end{enumerate}

\textbf{P1.18} \quad x^2 - 973 x + 1 = 0. \quad \text{(Round to three digits.)}
\textbf{P1.19} \quad x^2 - 57 x + 1 = 0. \quad \text{(Round to four digits.)}
\textbf{P1.20} \quad x^2 - 23 x + 1 = 0. \quad \text{(Round to four digits.)}

Problems P1.21 to P1.25 illustrate the use of the trapezoidal rule for numerical integration.
\begin{enumerate}
\item Approximate the given integral, using the basic trapezoidal rule with \( h = b - a \).
\item Approximate the integral, using the composite trapezoidal rule with \( h = \frac{b - a}{2} \).
\item Improve the approximation by acceleration, using results from parts a and b.
\end{enumerate}

\textbf{P1.21} \quad \text{Find } \int_0^2 \frac{1}{1 + x^2} \, dx.
\textbf{P1.22} \quad \text{Find } \int_0^{\pi/2} \sin(x) \, dx.
\textbf{P1.23} \quad \text{Find } \int_0^2 2^x \, dx.
\textbf{P1.24} \quad \text{Find } \int_0^2 e^{-x^2} \, dx.
\textbf{P1.25} \quad \text{Find } \int_0^{\pi/2} \frac{3}{1 + \sin(x)} \, dx.
Problems P1.26 to P1.30 illustrate the effect of the order of the equations in Gaussian elimination (with error).

a. Solve the equations in the order given; determine bounds on x and y.
   Graph the given equations, and the transformed second equation
b. Solve the equations in reverse order; determine bounds on x and y.
   Graph the given equations, and the transformed second equation
c. Compare the relative error in the values of x and y found in parts a and b.

\[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \]
\[ = \ldots ((a_n x + a_{n-1})x + a_{n-1})x \]
\[ + \ldots + a_1 x + a_0 \]

EXTEND YOUR UNDERSTANDING

**U1.1** For each of the Problems P1.6 to P1.10 for which the conditions of the theorem are not satisfied on [0, 1], investigate the following questions.

a. Is there a starting value of x for which the iterations do not converge? (Show this on the graph.)
b. Is there a subinterval on which the conditions are satisfied?

**U1.2** Use the Gerschgorin circle theorem to show that, for any matrix

\[ A = \begin{bmatrix} 1+2r & -r & 0 \\ -r & 1+2r & -r \\ 0 & -r & 1+2r \end{bmatrix} \]

any eigenvalue \( m \) satisfies \( |m| \geq 1 \), regardless of the value of \( r \). This result is used in Chapter 15. Extend the pattern to larger dimension matrices (e.g., tridiagonal, with \( 1+2r \) on the diagonal, and \( -r \) on the subdiagonal and superdiagonal).

**U1.3** Show that Horner’s method is equivalent to synthetic division.

**U1.4** Show that Horner’s method for evaluating polynomials can be viewed as rearranging the polynomial such that

**U1.5** Consider a very limited binary normalized floating-point system in which there are four bits to store the positive numbers. What exponents can be represented if 2 bits are used? What numbers can be represented if 2 bits are used for digits, and 2 for exponents? What numbers can be represented if 3 bits are used for digits and 1 for the exponents?

**U1.6** Find the positive numbers that can be represented with only 1 digit and exponents of 0 or 1 in a base-10 normalized floating-point system.