Chapter 5: Analysis of Deformation

Deformation:
When loads or forces are applied to a solid body this causes the body to change shape or geometry. Such a change is called “deformation” of that body. In fluids, the applied forces cause the fluids to “flow” instead of to “deform”.

Consider a piece of rubber in the form of a string of constant cross-section and length $L_0$, call this the “initial length”. If a force is applied to the two ends of the string, it will stretch, i.e. change shape or deform, to a length $L$. Call this length the “current length” produced by the applied forces. How would one quantify such a deformation in an objective way? One might say that to measure or quantify the deformation, we can use the difference in length, $\Delta L = L - L_0$, for this purpose. This measure, however, is not informative as the absolute value of $\Delta L$ is not as important as the relative ratio of $\Delta L$ to the original length $L_0$ say. For example, if $\Delta L$ is 1mm this can be seen as either too little if $L_0 = 1\text{m}$ or too much if $L_0 = 2\text{mm}$ say. This objective measure of deformation is what we call a “strain” measure. Hence, one measure of deformation or strain, call it $\varepsilon$, is

$$\varepsilon = \frac{L - L_0}{L_0} = \frac{L}{L_0} - 1 = \lambda - 1$$

where the ratio $L/L_0$ is termed the “stretch” and denoted by $\lambda$ here. Another measure of strain might be expressed as

$$\varepsilon' = \frac{L - L_0}{L} = 1 - \frac{L_0}{L} = 1 - \lambda^{-1}$$

where it might be elected to normalize by the current length $L$ instead. Notice that a prime was added to $\varepsilon$ here to distinguish it from the different strain measure given above first.

If the deformation, i.e. change of length, is infinitesimal, then the two measures of strain are the same essentially. For example, take $L = 1.00$ and $L_0 = 1.01$, then $\varepsilon = \varepsilon' = 0.01$. However, if $L = 2$ and $L_0 = 1$, then $\varepsilon = 1$ and $\varepsilon' = \frac{1}{2}$. and the two measures of strain give different results. It is to be noted at this juncture that other strain measures exists but will be left for future discussions.

There are two main modes of deformation. One of them is induced by normal stresses like pulling on or compressing a flat board of some thickness. See figure below. Here rectangles were sketched on the board (on its outside boundary) along its thickness before the force was applied. Such markers help us visualize the ensuing deformation after load application. Note in this particular example that the rectangles were stretched or elongated in one direction, the applied tensile stress direction, and compressed or shortened in the transverse or normal to that direction. Another mode of deformation is obtained by again marking a box of material, and looking at it from its frontal view, and then shearing the bottom and top faces of the box with shear forces. The ensuring deformation is shown on the right-hand side. In this type of deformation, the sides of each rectangle maintained original dimensions but changed interior angles. Any general deformation is composed of these two modes superposed.
Example of deformation due to tension or compression

Example of deformation due to shearing

**Strain in a material:**
Consider a small element of material of differential dimensions ABCD, originally a rectangle that is subjected to stresses and thus deforms to assume the shape A´B´C´D´. Such an element does not have to be stand-alone and can be considered as part of a larger body of material. Assume the analysis to be in 2D for simplicity. In 3D, the ideas are the same, and we thus can generalize the 2D results to 3D deformation.
Point A is displaced \( u \) units in the \( x \)-direction and \( v \) units in the \( y \)-direction. In the above figure, everything is exaggerated, e.g. the size of the element and the displacements. Here we are only concerned about small displacements or deformations of the infinitesimal element. Notice that the deformed shape \( A'B'C'D' \) can be obtained from ABCD by three operations in sequence:

1. Rigid body translation to \( A''''B''''C''''D'''' \)
2. Uniform stretching of \( A''''B''''C''''D'''' \) in the \( x \) and \( y \) directions to \( A'''''B''''C''''D'''' \) which changes the lengths of element’s sides
3. Uniform shearing of the element to \( A'B'C'D' \). This shearing does not change the length of the sides of \( A'''''B''''C''''D'''' \) but changes the angles between two originally perpendicular sides of the element.

During step 1, \( AB = A''''B'''' = DC = D'''C''' \), and \( AD = A''''D'''' = BC = B'''C''' \) which preserves the side lengths but distorts or changes the angle between any two originally perpendicular sides of the element. Notice that this step does not have to be purely rigid body translation. It can also be a rigid body rotation that also preserves the original side lengths. However, for simplicity of illustration only the translation is shown. Notice that during step 2, the side lengths of \( A'B'C'D' \) change. Therefore, \( A''''B'''' \neq A''''B'''' \), \( D'''C''' \neq D'''C''' \), and \( A''''D'''' \neq A''''D'''' \), \( B'''C''' \neq B'''C''' \). Finally during step 3, the side lengths stay the same (i.e. \( A''''B'''' = A''''B'''' \), \( D'''C''' = D'''C''' \), \( A''''D'''' = A''''D'''' \), \( B'''C''' = B'''C''' \)). However, during this step, the initial right angles change. For example, the angle \( \perp D'''A'''B''' \) becomes \( \perp D'A'B' \) which is less than \( 90^\circ \) by an amount equal to \( \alpha + \alpha' \). Similarly, for the angle \( \perp B'''C'''D''' \). However, the angle \( \perp A'B'C' \) is greater than \( 90^\circ \) by an amount equal to \( \alpha + \alpha' \). Similarly for \( \perp A'D'C' \). Notice that in step 1, the rectangle ABCD simply translates to \( A''''B''''C''''D'''' \).

Before we carry on, notice a couple of things. First, the displacements of point A (the lower left corner of the element) in the \( x \) and \( y \)-directions are given by \( u \) and \( v \) (or equivalently by \( u_x \) and \( u_y \) or by \( u_1 \) and \( u_2 \), respectively. Since we randomly picked a
material element, hence the displacements of point A (the lower left corner of the material element) will depend on the spatial position of this element or point within the larger continuum or body that contains it, i.e. \( u = u(x,y) \) and \( v = v(x,y) \). In general, \( u = u(x,y,z) \), \( v = v(x,y,z) \) and there is a third component of displacement \( w \) perpendicular to the plane of \( u \) and \( v \) which is also a function of the three spatial coordinates, i.e. \( w = w(x,y,z) \).

For example, if the displacements of point A in a body, or in an element in a body, is known as \( u = u(x,y) \) and \( v = v(x,y) \), then the displacements of any nearby point P (like points C, B or D) can be found from \( u = u(x,y) \) and \( v = v(x,y) \) using Taylor series expansion as follows:

\[
\begin{align*}
  u^P & = u + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial^2 u}{\partial x^2} (dx)^2 + \frac{\partial^2 u}{\partial y^2} (dy)^2 + \frac{\partial^2 u}{\partial x \partial y} dx dy + \cdots \\
  v^P & = v + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial^2 v}{\partial x^2} (dx)^2 + \frac{\partial^2 v}{\partial y^2} (dy)^2 + \frac{\partial^2 v}{\partial x \partial y} dx dy + \cdots 
\end{align*}
\]

Similarly,

\[
\begin{align*}
  u^P & = u + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial^2 u}{\partial x^2} (dx)^2 + \frac{\partial^2 u}{\partial y^2} (dy)^2 + \frac{\partial^2 u}{\partial x \partial y} dx dy + \cdots \\
  v^P & = v + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial^2 v}{\partial x^2} (dx)^2 + \frac{\partial^2 v}{\partial y^2} (dy)^2 + \frac{\partial^2 v}{\partial x \partial y} dx dy + \cdots 
\end{align*}
\]

where \( dx = x^P - x^A = x^P - x \), \( dy = y^P - y^A = y^P - y \), \( u^P \) and \( v^P \) represent the displacements of any point P in the material knowing the displacements \( u \) and \( v \) of another point A (here we are assuming continuous displacement distributions or function in the continuum, otherwise it is not be possible to take the derivatives).

Considering very small variations, i.e. gradients, of displacement and that points A and P are very close from each other to begin with, we can drop the higher order derivatives in the last two equations and write (or assume to first order accuracy):

\[
\begin{align*}
  u^P & \equiv u + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \\
  v^P & \equiv v + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy 
\end{align*}
\]

If point P happens to be point B, then we have

\[
\begin{align*}
  u^B & \equiv u + \frac{\partial u}{\partial x} dx \quad \text{(since } dy = 0 \text{ here)} \\
  v^B & \equiv v + \frac{\partial v}{\partial x} dx \quad \text{(since } dy = 0 \text{ here)}
\end{align*}
\]

Also,

\[
\begin{align*}
  u^D & \equiv u + \frac{\partial u}{\partial y} dy \quad \text{(since } dx = 0 \text{ here)} \\
  v^D & \equiv v + \frac{\partial v}{\partial y} dy \quad \text{(since } dx = 0 \text{ here)}
\end{align*}
\]

From this last figure, we see that we can define two types of deformation measures or strain to which the element is subjected. One type of strain is associated with stretching of element ABCD (or alternatively A‘’B‘’C‘’D‘’). This is step 2. This type of strain which involves stretching or contracting of a material element is termed “extensional” or “normal strains”. If the strains are extensional then they are called “tensile” strains, if
they are contractional then they are called “compressive” strains. The other type of strain is associated with shearing of element A′′′B′′′C′′′D′′′ by changing the angles between element sides while preserving their lengths (e.g. from A′′′B′′′C′′′D′′′ to A′B′C′D′). The shearing preserves the lengths of element sides but changes the angles between its sides. This deformation describes what is called “shearing” or “shear strain”.

The above two definitions of strain can be mathematically formulated or stated as follows.

Normal, extensional or stretching strains (also called tensile or compressive stresses) in the x-direction:

\[ e_{xx} = \frac{A''B'' - AB}{AB} = \frac{A''B'' - dx}{dx} - 1 \]

The normal strain in the y-direction:

\[ e_{yy} = \frac{A''D'' - AD}{AD} = \frac{A''D'' - dy}{dy} - 1 \]

Notice that the above definition of normal strains produces unitless or dimensionless strains since the changes in length are relative or normalized to the original unstretched lengths. Similar to stress definition, the first subscript indicates the direction of the face or the plane that is normal to the extended line (AB or AD in this case). The second subscript indicates the direction of the extension itself.

To define shear strain, we can write:

\[ \gamma_{xy} = \frac{\pi}{2} - \beta = \text{change in angle (in radians)} = \alpha + \alpha' \]

, where the first subscript indicates the face normal direction, and the second subscript indicates the shearing direction.

Notice in the above definitions of strain that if we are only assuming small changes in displacements and small changes of angles, we should then expect that the resulting strains be much less than unity (i.e. \( e_{xx} << 1 \), \( e_{yy} << 1 \), and \( \gamma_{xy} << 1 \))

The shearing strain \( \gamma_{xy} \) is called the “engineering shear strain” and is equal to \( 2e_{xy} \), where \( e_{xy} \) is called the tensorial shear strain. \( e_{xy} \) is given by \( e_{xy} = \frac{1}{2} (\alpha + \alpha') \) which represents the average angle change between \( \alpha \) (or angle B′′A′′B′) and \( \alpha' \) (or angle D′′A′′D′).

From the definitions of strain, we can write:

\[ (A'B')^2 = (A''B'')^2 = (dx(1 + e_{xx}))^2 \]

But from the last sketch,

\[ (A'B')^2 = (dx + \frac{\partial u}{\partial x} dx)^2 + (\frac{\partial v}{\partial x} dx)^2 \]

Equate the last two equations, we get:

\[ (1 + 2e_{xx} + e_{xx}^2)(dx)^2 = (dx)^2 + 2\left( \frac{\partial u}{\partial x} \right)(dx)^2 + \left( \frac{\partial u}{\partial x} \right)^2 (dx)^2 + \left( \frac{\partial v}{\partial x} \right)^2 (dx)^2 \]

\[ (*) \]
Assuming very small deformation (i.e. displacements and gradients of displacement $\ll 1$), we can then drop the squared or higher-order terms from the last equation to get

$$e_{xx} = \frac{\partial u}{\partial x} \quad (A)$$

Similarly, we can find

$$e_{yy} = \frac{\partial v}{\partial y} \quad (B)$$

Also, from the previous sketch

$$\alpha = \frac{\partial v}{\partial x} \frac{dx}{1 + \frac{\partial u}{\partial x}}$$

Assuming $\frac{\partial u}{\partial x} \ll 1 \Rightarrow 1 + \frac{\partial u}{\partial x} \approx 1$

Hence, $\alpha = \frac{\partial v}{\partial x}$

Similarly, $\alpha' = \frac{\partial u}{\partial y}$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (C)$$

, where the partial derivatives are positive if A’’’ B’’’ and A’’’ D’’’ rotate inward as shown in the sketch.

Equations (A), (B) and (C) are called the strain-displacement relations for “small strain theory” (also called “infinitesimal strain theory”). In some texts, this is also called “small displacement theory”.

Notice that using (C), the tensorial shear strain (or simply shear strain) is given by

$$e_{xy} = \frac{\gamma_{xy}}{2} = \frac{\alpha + \alpha'}{2} = \frac{1}{2} (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \quad (D)$$

which is simply the average of the two angles $\alpha$ and $\alpha'$.

So far we have dealt with 2D or planar strains (equations (A)-(D)). In 3D, the strains become:

$$e_{xx} = \frac{\partial u}{\partial x}$$

$$e_{yy} = \frac{\partial v}{\partial y}$$

$$e_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
\[ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \]
\[ \gamma_{yz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial y} \]

The last equations for strain can be written more concisely using index (indicial) or tensor notation as:
\[ e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i = 1 \ldots 3, \quad j = 1 \ldots 3 \]

From the last notation, we notice that
\[ e_{ii} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} \right) = e_{ij} \]

Now, let's go back to (*), if we only drop \( e_{xx} \) from that equation, we get:
\[ e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 + \frac{\partial v}{\partial x} \left( \frac{\partial v}{\partial x} \right)^2 \]
\[ e_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\partial v}{\partial y} \left( \frac{\partial v}{\partial y} \right)^2 \]

and
\[ e_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial x} \right) \]

The above strain equations are called the strain-displacement for “finite strains” or “large strains”.

The 3-D version of the above strain equation is:
\[ e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 + \frac{\partial v}{\partial x} \left( \frac{\partial v}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \left( \frac{\partial w}{\partial x} \right)^2 \]
\[ e_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\partial v}{\partial y} \left( \frac{\partial v}{\partial y} \right)^2 + \frac{\partial w}{\partial y} \left( \frac{\partial w}{\partial y} \right)^2 \]
\[ e_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2} \left( \frac{\partial u}{\partial z} \right)^2 + \frac{\partial v}{\partial z} \left( \frac{\partial v}{\partial z} \right)^2 + \frac{\partial w}{\partial z} \left( \frac{\partial w}{\partial z} \right)^2 \]
\[ e_{xy} = e_{yx} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial x} \right) \]
\[ e_{xz} = e_{zx} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial w}{\partial z} \right) \]
\[ e_{yz} = e_{zy} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial z} \right) \]
The last equations can be written concisely as:

\[ e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) \]

It is important to note that in developing the strain-displacement relations above, no assumption was made regarding the constitution of the material (whether it is isotropic or anisotropic, homogeneous or inhomogeneous). The whole derivation was solely carried out on geometric grounds or arguments.

Notice also that the different strains can be considered to be the components of a two-dimensional array, or second-rank tensor in other words, \( e \), such that

\[
\begin{bmatrix}
  e_{xx} & e_{xy} & e_{xz} \\
  e_{yx} & e_{yy} & e_{yz} \\
  e_{zx} & e_{zy} & e_{zz}
\end{bmatrix}
\]

for 3-D

And

\[
\begin{bmatrix}
  e_{xx} & e_{xy} \\
  e_{yx} & e_{yy}
\end{bmatrix}
\]

for 2-D

And just as we solved for the “principal stresses” if given a stress tensor \( \sigma \), we can solve for the “principal strains” if given a strain tensor \( e \). This is done by solving for the roots of the characteristic equation that results from

\[
\begin{vmatrix}
  e_{xx} - \epsilon & e_{xy} & e_{xz} \\
  e_{yx} & e_{yy} - \epsilon & e_{yz} \\
  e_{zx} & e_{zy} & e_{zz} - \epsilon
\end{vmatrix} = 0
\]

The resulting characteristic equation can be written as:

\[ e^3 - e^2 \bar{T}_1 - e \bar{T}_2 - \bar{T}_3 = 0 \]

where \( \bar{T}_1, \bar{T}_2 \) and \( \bar{T}_3 \) are called the invariants of the strain tensor and are given by

\[
\bar{T}_1 = e_{xx} + e_{yy} + e_{zz} = e_{ij}
\]

\[
\bar{T}_2 = -\begin{vmatrix}
  e_{xx} & e_{xy} \\
  e_{yx} & e_{yy}
\end{vmatrix} - \begin{vmatrix}
  e_{xx} & e_{xz} \\
  e_{zx} & e_{zz}
\end{vmatrix} - \begin{vmatrix}
  e_{yy} & e_{yz} \\
  e_{zy} & e_{zz}
\end{vmatrix} = e_{xy}^2 + e_{xz}^2 + e_{yz}^2 - e_{xx}e_{yy} - e_{xx}e_{zz} - e_{yy}e_{zz}
\]

\[
\bar{T}_3 = \begin{vmatrix}
  e_{xx} & e_{xy} & e_{xz} \\
  e_{yx} & e_{yy} & e_{yz} \\
  e_{zx} & e_{zy} & e_{zz}
\end{vmatrix} = |e|
\]

We can also find the “principal strain directions” just as we’ve done for the stresses.

Now since strain \( e \) or \( e_{ij} \) is a second-rank tensor, it transforms accordingly:

\[ e'_{p'q'} = \beta_{p'j} \beta_{q'i} e_{ij} \]

Or

\[ e_{ij} = \beta_{ip} \beta_{jq} e'_{p'q'} \]
which means that if one knows the strain tensor in one orthogonal coordinate system, it is possible to find the strain tensor with respect to another orthogonal coordinate system if the rotation or transformation matrix between the two is known or can be established at least.

**The Strain in a material: Another Look**

The deformation in a material can be looked at differently than what was presented above. Consider a small material volume or particle that initially was located at point \( P \) with coordinates \((a_1, a_2, a_3)\), this is called the “original configuration”. After deformation, or at a given time of consideration during the deformation, the particle moved to point \( Q \) with coordinates \((x_1, x_1, x_3)\). This is called the “current” or “final configuration”. The vector \( \vec{PQ} \) is called the displacement vector. It is also denoted by the symbol \( u \). Hence,

\[
iii \ axu
\]

If the displacement vector of every particle in the original configuration is known, then one can construct the deformed body, i.e. the current or final configuration, from such knowledge. Moreover, the coordinates, i.e. the \( x_i \)’s, of every point in the deformed body must have depended on the location of that point in the body prior to deformation, i.e. on \((a_1, a_2, a_3)\). This is because there is a one-to-one correspondence between these two locations. Mathematically, this is stated as

\[
\),,( 321 \aa a x =
\]

Vice versa, it can be easily seen that the inverse relationship should also exist, i.e. that there is a reverse mapping that maps the current configuration onto the original one. Mathematically, this is stated as

\[
\),,( 321 \aaa a x =
\]

Hence, the displacement vector can be written as

\[
u_i(a_1,a_2,a_3) = x_i(a_1,a_2,a_3) - a_i
\]

or as

\[
u_i(x_1,x_2,x_3) = x_i - a_i(x_1,x_2,x_3)
\]
Consider an infinitesimal line element connecting the point \( P(a_1, a_2, a_3) \) to a neighboring point \( P'(a_1+da_1, a_2+da_2, a_3+da_3) \). The square of the length \( ds_0 \) of \( PP' \) in the original undeformed configuration is given by

\[
ds_0^2 = da_1^2 + da_2^2 + da_3^2 = da_i da_i
\]

When \( P \) and \( P' \) are deformed to point \( Q(x_1, x_2, x_3) \) and \( Q'(x_1+dx_1, x_2+dx_2, x_3+dx_3) \), respectively, the square of the length \( ds \) of the element \( QQ' \) in the current or final deformed configuration is given by

\[
ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_i dx_i
\]

But using the equations above, we know that

\[
dx_i = \frac{\partial x_i}{\partial a_j} da_j, \quad da_i = \frac{\partial a_i}{\partial x_j} dx_j
\]

The last two equations can be re-written using the Kronecker delta as

\[
ds_0^2 = \delta_{\alpha\beta} da_\alpha da_\beta = \delta_{\alpha\beta} \frac{\partial a_\alpha}{\partial x_i} \frac{\partial a_\beta}{\partial x_j} dx_i dx_j
\]

\[
ds^2 = \delta_{\alpha\beta} dx_\alpha dx_\beta = \delta_{\alpha\beta} \frac{\partial x_\alpha}{\partial a_i} \frac{\partial x_\beta}{\partial a_j} da_i da_j
\]

Note now that one can now write...
\[ ds^2 - ds_0^2 = \delta_{\alpha\beta} \frac{\partial x^\alpha}{\partial a_i} \frac{\partial x^\beta}{\partial a_j} da_i da_j - \delta_{ij} da_i da_j = \left( \delta_{\alpha\beta} \frac{\partial x^\alpha}{\partial a_i} \frac{\partial x^\beta}{\partial a_j} - \delta_{ij} \right) da_i da_j \]

Or
\[ ds^2 - ds_0^2 = \delta_{ij} dx_i dx_j - \delta_{\alpha\beta} \frac{\partial a^\alpha}{\partial x_i} \frac{\partial a^\beta}{\partial x_j} dx_i dx_j = \left( \delta_{ij} - \delta_{\alpha\beta} \frac{\partial a^\alpha}{\partial x_i} \frac{\partial a^\beta}{\partial x_j} \right) dx_i dx_j \]

We define here two different “strain tensors”
\[ E_{ij} = \frac{1}{2} \left( \delta_{\alpha\beta} \frac{\partial x^\alpha}{\partial a_i} \frac{\partial x^\beta}{\partial a_j} - \delta_{ij} \right) \]
\[ e_{ij} = \frac{1}{2} \left( \delta_{ij} - \delta_{\alpha\beta} \frac{\partial a^\alpha}{\partial x_i} \frac{\partial a^\beta}{\partial x_j} \right) \]

So that
\[ ds^2 - ds_0^2 = 2E_{ij} da_i da_j \]
\[ ds^2 - ds_0^2 = 2e_{ij} dx_i dx_j \]

The strain tensor \( E_{ij} \) was introduced by Green and St.-Venant and is called “Green’s strain tensor”. The strain tensor \( e_{ij} \) was introduced by Cauchy for infinitesimal strains (small-displacement theory) and by Almansi and Hamel for finite strains (i.e. finite-displacement theory). \( E_{ij} \) is often referred to as the “Lagrangian strain tensor” and \( e_{ij} \) is often referred to as the “Eulerian strain tensor”, in analogy to terminology in hydrology or fluid mechanics, as the derivatives are taking with respect to the old or initial configurations in \( E_{ij} \) and with respect to the current or final configuration in \( e_{ij} \).

As can be seen from the last equations, the tensors \( E_{ij} \) and \( e_{ij} \) are symmetric, i.e.
\[ E_{ij} = E_{ji}, \quad e_{ij} = e_{ji} \]

Another observation about the above equations is that if the line length of \( QQ' \) remains unchanged from \( PP' \) then we must have \( ds^2 = ds_0^2 \) which implies that \( E_{ij} = e_{ij} = 0 \). If the length of small lines ALL remain unchanged during the displacement then the body must have experienced rigid body motion and no deformation. Hence, a necessary indeed sufficient condition for rigid body motion definition is that the strain tensor \( E_{ij} \) or \( e_{ij} \) must be zero when evaluated at every point in the body.
Strain Components in terms of Displacement:

From before, we know that we can write

\[ u_\alpha = x_\alpha - a_\alpha \quad (\alpha = 1,2,3) \]

This last equation can be re-arranged to be

\[ x_\alpha = u_\alpha + a_\alpha \quad \text{or} \quad a_\alpha = x_\alpha - u_\alpha \]

Taking derivatives, we can write

\[ \frac{\partial x_\alpha}{\partial a_i} = \frac{\partial u_\alpha}{\partial a_i} + \delta_{\alpha i} \quad \text{and} \quad \frac{\partial a_\alpha}{\partial x_i} = \delta_{\alpha i} - \frac{\partial u_\alpha}{\partial x_i} \]

Based on this, the strain tensors can be written as

\[
E_{ij} = \frac{1}{2} \left( \delta_{\alpha \beta} \left( \frac{\partial u_\alpha}{\partial a_i} + \delta_{\alpha i} \right) \left( \frac{\partial u_\beta}{\partial a_j} + \delta_{\beta j} \right) - \delta_{ij} \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial u_j}{\partial a_i} + \frac{\partial u_i}{\partial a_j} + \frac{\partial u_\alpha}{\partial a_i} \frac{\partial u_\alpha}{\partial a_j} \right)
\]
\[ e_{ij} = \frac{1}{2} \left( \delta_{ij} - \delta_{\alpha\beta} \left( \frac{\partial u_\alpha}{\partial x_i} - \frac{\partial u_\alpha}{\partial x_j} \right) \right) \]
\[ = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_\alpha}{\partial x_i} \frac{\partial u_\alpha}{\partial x_j} \right) \]

Replacing \( x_1, x_1, x_3 \) with \( x, y, z \), and \( a_1, a_2, a_3 \) with \( a, b, c \), and \( u_1, u_2, u_3 \) with \( u, v, w \), we can write the above as

\[ E_{aa} = \frac{\partial u}{\partial a} + \frac{1}{2} \left( \left( \frac{\partial u}{\partial a} \right)^2 + \left( \frac{\partial v}{\partial a} \right)^2 + \left( \frac{\partial w}{\partial a} \right)^2 \right) \]

\[ E_{bb} = \frac{\partial v}{\partial b} + \frac{1}{2} \left( \left( \frac{\partial u}{\partial b} \right)^2 + \left( \frac{\partial v}{\partial b} \right)^2 + \left( \frac{\partial w}{\partial b} \right)^2 \right) \]

\[ E_{cc} = \frac{\partial w}{\partial c} + \frac{1}{2} \left( \left( \frac{\partial u}{\partial c} \right)^2 + \left( \frac{\partial v}{\partial c} \right)^2 + \left( \frac{\partial w}{\partial c} \right)^2 \right) \]

\[ E_{ab} = \frac{1}{2} \left( \frac{\partial u}{\partial b} + \frac{\partial v}{\partial a} + \frac{\partial u}{\partial a} \frac{\partial u}{\partial b} + \frac{\partial v}{\partial a} \frac{\partial v}{\partial b} + \frac{\partial w}{\partial a} \frac{\partial w}{\partial b} \right) \]

\[ E_{ac} = \frac{1}{2} \left( \frac{\partial u}{\partial c} + \frac{\partial v}{\partial a} + \frac{\partial u}{\partial a} \frac{\partial u}{\partial c} + \frac{\partial v}{\partial a} \frac{\partial v}{\partial c} + \frac{\partial w}{\partial a} \frac{\partial w}{\partial c} \right) \]

\[ E_{bc} = \frac{1}{2} \left( \frac{\partial v}{\partial b} + \frac{\partial w}{\partial c} + \frac{\partial v}{\partial b} \frac{\partial v}{\partial c} + \frac{\partial w}{\partial b} \frac{\partial w}{\partial c} \right) \]

\[ e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right) \]

\[ e_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right) \]

\[ e_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2} \left( \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right) \]

\[ e_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \left( \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right) \]

\[ e_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \left( \frac{\partial u}{\partial x} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial z} \right) \right) \]

\[ e_{yz} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \left( \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \right) \right) \]
If the relative difference between the displacements of one point in the material and another neighboring one, i.e. the gradient of displacement, is very small then the products of these gradients is much smaller compared to the gradients themselves and can be neglected. In this case \( e_{ij} \) reduces to the “Cauchy infinitesimal strain tensor”

\[
e_{ij} = E_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)
\]

In the case of infinitesimal displacement, which means that \((x_1, x_1, x_3)\) is essentially \((a_1, a_2, a_3)\), the distinction between the Lagrangian and Eulerian strain tensors disappears since it then makes no difference if derivatives are carried out with respect to the original or deformed configuration.

**Rigid Body Rotation during Displacement:**

Consider again points \( P \) and \( P' \) from before. Those two points are separated by a small differential vector \( dx \), where point \( P \) lies at \( x \) and point \( P' \) lies at \( x + dx \), and for this discussion lets assume infinitesimal displacements such that there is not much difference between the initial and final configurations. For the components of \( du \), we can write

\[
du_i = \frac{\partial u_i}{\partial x_j} dx_j
\]

We can re-write the last equation as

\[
du_i = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dx_j + \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) dx_j
\]

\[
\varepsilon_{ij} \quad \omega_{ji}
\]

If the infinitesimal strain tensor, \( \varepsilon_{ij} \), tends to zero, i.e. there is no deformation in the body, then the difference in the displacement of these two points is due to rigid-body motion by definition. Hence, the tensor \( \omega_{ji} \) must be related to the rotation of that material element or particle associated with point \( P \) and its immediate neighborhood. This tensor is called the “rotation tensor” of the displacement field \( u_i \).

Therefore, in this case, we can write the last equation as

\[
du_i = \omega_{ji} dx_j
\]

Note, however, that the rotation tensor is anti-symmetric, i.e.

\[
\omega_{ij} = \omega_{ji}
\]

For any such anti-symmetric tensor we can always find what is called a “dual vector” associated with this tensor. We build this vector as

\[
\omega_k = \frac{1}{2} \varepsilon_{kij} \omega_{ij}
\]

If this is a dual vector, i.e. one that is derived from the tensor and associated with it, we should also be able to get the inverse relationship as well, i.e. express the tensor in terms of the dual vector. To show that, multiply the last equation by \( \varepsilon_{ijk} \)

\[
\varepsilon_{ijk} \omega_k = \frac{1}{2} \varepsilon_{ijk} \varepsilon_{kmn} \omega_{mn} = \frac{1}{2} \varepsilon_{kij} \varepsilon_{kmn} \omega_{mn} = \frac{1}{2} (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \omega_{mn} = \frac{1}{2} (\omega_{ij} - \omega_{ji}) = \omega_{ij}
\]
Hence
\[ \omega_{ij} = \epsilon_{ijk} \omega_k \]
The vector \( \omega \) above is called the “rotation vector” of the displacement field \( u_i \).

Based on all of the above, we can now write
\[
\begin{align*}
\text{du}_i &= \omega_{ji}dx_j = -\omega_{ij}dx_j = -\epsilon_{ijk} \omega_k dx_j = \epsilon_{ijk} \omega_k dx_j = (\omega \times dx)_i \\
\end{align*}
\]
Therefore
\[
\text{du} = \omega \times dx
\]

**Finite Strain Components:**
For large strains it is not as easy as for small strains to give a geometric interpretation of the deformation. We do so here for one component of strain \( E_{11} \).

Consider rectangular coordinates and a line element of material that is \( \text{d}a \) before deformation ensues. Let the components of \( \text{d}a \) be \( da_1 = ds_0, da_2 = 0, da_3 = 0 \). Let the extension of the this element \( E_1 \) be defined by
\[
E_1 = \frac{ds - ds_0}{ds_0}
\]
or
\[
ds = (1 + E_1)ds_0 \Rightarrow ds^2 = (1 + E_1)^2 ds_0^2 = (1 + E_1)^2 (da_1)^2
\]
From before we know that
\[
ds^2 - ds_0^2 = 2E_{ij}da_ida_j = 2E_{11}(da_1)^2 \Rightarrow ds^2 = ds_0^2 + 2E_{11}(da_1)^2 = (da_1)^2 + 2E_{11}(da_1)^2
\]
From the last two equations, we can write
\[
2E_{11} = (1 + E_1)^2 - 1
\]
This last equation gives the meaning of \( E_{11} \) in terms of \( E_1 \). Alternately, we can write
\[
E_1 = \sqrt{1 + 2E_{11} - 1}
\]
If \( E_{11} \) is small compared to 1, i.e. we have the case of small strains, the last equation reduces to
\[
E_1 = E_{11}
\]
Which is the geometric interpretation of strain in the case of small strains.

**Infinitesimal Strain Components in Polar Coordinates:**
The relationships between rectangular Cartesian coordinates and polar coordinates are given by
\[
\begin{align*}
x &= r \cos \theta, \quad \theta = \tan^{-1} \frac{y}{x}, \\
y &= r \sin \theta, \quad r^2 = x^2 + y^2, \\
z &= z
\end{align*}
\]
\[
\frac{\partial r}{\partial x} = \frac{x}{r} = \cos \theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta,
\]
\[
\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\sin \theta, \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \cos \theta
\]

From the theory of partial differentiation, we can write
\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \frac{r}{\partial \theta} \frac{\partial}{\partial \theta},
\]
\[
\frac{\partial}{\partial y} = \frac{\partial}{\partial y} + \frac{r}{\partial \theta} \frac{\partial}{\partial \theta}
\]

The last derivatives can be transformed into derivatives with respect to \( x \) and \( y \)
\[
\frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta},
\]
\[
\frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta}
\]

In polar coordinates, the displacement vector \( \mathbf{u} \) has components \( u_r, u_\theta, u_z \) as shown in the figure.

These components can be resolved into the \( x, y \) and \( z \) directions of the Cartesian coordinate system to give the \( u_x, u_y, \) and \( u_z \) components. The relationship between the Cartesian and Polar displacement components is given by
\[
u'_i = \beta_{ij} u_j
\]
or
\[
u_i = \beta_{ji} u'_j
\]
In expanded form, the above equations can be written as
\[ u_x = u_r \cos \theta - u_\theta \sin \theta \]
\[ u_y = u_r \sin \theta + u_\theta \cos \theta \]
\[ u_z = u_z \]

The strain tensor in polar coordinates is given by
\[
\mathbf{e} = \begin{bmatrix}
  e_{rr} & e_{r\theta} & e_{rz} \\
  e_{\theta r} & e_{\theta\theta} & e_{\theta z} \\
  e_{zr} & e_{z\theta} & e_{zz}
\end{bmatrix}
\]

The transformation between strains in Cartesian coordinates and in polar coordinates is governed by
\[ e_{\alpha\beta}' = \beta_{\alpha\alpha} e_{\alpha\beta} \]
Which, in expanded form, can be written as
\[ e_{rr}' = e_{xx} \cos^2 \theta + e_{yy} \sin^2 \theta + 2 \cos \theta \sin \theta e_{xy} \]
\[ e_{\theta\theta}' = e_{xx} \sin^2 \theta + e_{yy} \cos^2 \theta - 2 \cos \theta \sin \theta e_{xy} \]
\[ e_{zz}' = e_{zz} \]
\[ e_{r\theta}' = (e_{yy} - e_{xx}) \sin \theta \cos \theta + (2 \cos^2 \theta - 1) e_{xy} \]
\[ e_{xz}' = e_{xz} \cos \theta + e_{yz} \sin \theta \]
\[ e_{tk}' = -e_{xz} \sin \theta + e_{yz} \cos \theta \]

Using the definitions
\[
e_{xx} = \frac{\partial u_x}{\partial x}
\]
\[
e_{yy} = \frac{\partial u_y}{\partial y}
\]
\[
e_{zz} = \frac{\partial u_z}{\partial z}
\]
\[
e_{xy} = \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)
\]
\[
e_{xz} = \frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)
\]
\[
e_{yz} = \frac{1}{2} \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right)
\]
one can now combine all of the previous equations to get
\[ e_{rr}' = \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \]
The last equations are the expressions of the strain tensor components in a polar coordinate system (using polar coordinates and displacement components in polar coordinates).

Of course, an alternative to deriving the strain components in polar coordinates is to resort to graphical interpretation of the different strain components. This is done in the textbook and is not repeated here. The important thing is that both ways give exactly the same final results.

**Strain-Compatibility Relations:**

If we are given the displacements $u$, $v$ and $w$ as a function of spatial coordinates, then we can derive from them the different strain components using the displacement-strain relationships we’ve learned before. However, if we are given the strain components (six of them in the 3-D case), we can not in general integrate the strains (using the displacement-strain relationships) to uniquely determine the 3 displacement components $u$, $v$ and $w$.

To uniquely determine the strains, we need to have more relations among the strain components. These relations are termed the “strain compatibility relations” and they are now illustrated for the 2-D case. Consider a case of “plane strain” where all non-vanishing strains lie in the $xy$-plane and any strains associated with the normal to the plane are all zeroes. This state of strain can happen if $u = u(x,y)$, $v = v(x,y)$ and $w = \text{constant}$. For here, we get (assuming infinitesimal strains)

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$e_{zz} = e_{xz} = e_{yz} = 0$$

In this case, there is a relation (a “strain compatibility relation”) that exists between the planar non-vanishing strains. We can obtain this relation by differentiation as follows:

$$\frac{\partial}{\partial y} e_{xx} = \frac{\partial^3 u}{\partial x \partial y^2}, \quad \frac{\partial^2}{\partial x^2} e_{yy} = \frac{\partial^3 v}{\partial x^2 \partial y}, \quad 2 \frac{\partial^2}{\partial x \partial y} e_{xy} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

Now, notice that the from the last equations, we can write

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2 \frac{\partial^2 e_{xy}}{\partial x \partial y}$$
This is a relation among three infinitesimal strain components. It is also given for 2D deformation and thus called “strain compatibility for plane strain”. In the general 3D case, six second-order equations or conditions exist among six components of strain.

To reiterate, if one is given the displacements \((u,v,w)\) in terms of the spatial coordinates, then one can simply substitute them into the displacement-strain relationships, i.e. differentiate and find the strains using small strain theory. As long as the displacements represent continuous functions of the spatial coordinates, then the strain will also be continuous functions that satisfy the compatibility equations.

However, if one instead is given the strains as a function of the spatial coordinates, one has to make sure first that these functions satisfy the strain-compatibility equations, otherwise this is not a permissible strain state distribution of the material, and one can not uniquely determine the displacements out of the given strains.

The physical meaning of “incompatible” strains is that the given strain distributions prescribes a condition of either tearing or overlapping in the material (i.e. a condition of discontinuity in the displacements).

**Example 1 on strains:**
Consider a square plate of unit size deformed as shown in the figure. Determine the strain components.
Note: This state of deformation is termed “simple shear”

**Solution:**
The mapping of the material point coordinates before deformation \((a_1,a_2,a_3)\) and after deformation \((x_1,x_2,x_3)\) is given by
\[
x_1 = a_1 + \frac{1}{\sqrt{3}} a_2, \quad x_2 = a_2, \quad x_3 = a_3
\]
Where \(1/\sqrt{3} = \tan30^\circ\)
Inversely, we can write
\[
a_1 = x_1 - \frac{1}{\sqrt{3}} x_2, \quad a_2 = x_2, \quad a_3 = x_3
\]
Hence, the displacement components are

\[ u_1 = x_1 - a_1 = \frac{1}{\sqrt{3}}, \quad a_2 = \frac{1}{\sqrt{3}} x_2, \quad u_2 = x_2 - a_2 = 0, \quad u_3 = x_3 - a_3 = 0 \]

The strain components are

\[ E_{11} = E_{33} = E_{13} = E_{23} \]
\[ E_{22} = 1/6 \]
\[ E_{12} = 1/2\sqrt{3} \]
\[ e_{11} = e_{33} = e_{13} = e_{23} \]
\[ e_{22} = -1/6 \]
\[ e_{12} = 1/2\sqrt{3} \]

Example 2 on strains:
ABCD is a unit square in the \( xy \)-plane (see figure) and is part of a large deformable body. If the strain in the body is small, uniform and given by

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 1 & 0 \\
3 & 0 & 2
\end{bmatrix} \times 10^{-3}
\]

What is the change in length of the line AC due to deformation?

Solution:
The original line length of AC before deformation is \( ds_0 = \sqrt{2} = 1.41421356 \)
The new line length of AC is \( ds \), the difference between the squares of these two lengths is given by:

\[ ds^2 - ds_0^2 = 2e_{ij}dx_idx_j \]

\[ ds^2 - ds_0^2 = 2[e_{11}dx_1dx_1 + e_{22}dx_2dx_2 + e_{33}dx_3dx_3] \]
\[ ds^2 - ds_0^2 = 2[e_{11}dx_1dx_1 + e_{12}dx_1dx_2 + e_{13}dx_1dx_3 + e_{22}dx_2dx_2] \]
\[ ds^2 - ds_0^2 = 2[e_{11}(dx_1)^2 + 2e_{12}dx_1dx_2 + e_{22}(dx_2)^2] \]
\[ dx_1 = dx_2 = 1 \]
\[ \Rightarrow ds^2 - 2 = 2[1 \times (1)^2 + 2 \times 2 \times 1 \times 1 + 1 \times (1)^2] \times 10^{-3} \]
\[ ds^2 = 2 + 0.012 = 2.012 \Rightarrow ds = 1.41845 \]
\[ ds - ds_0 = \text{change in length} = 0.004236 \]
Example 3 on strains:
A displacement field is defined by
\[ u = u_1 = -C x_2 + B x_3, \quad v = u_2 = C x_1 - A x_3, \quad w = u_3 = -B x_1 + A x_2 \]
Show that this field represents pure rotation and no deformation.

Solution:
For deformation to happen, we need to have non-zero strain \( e_{ij} \) component(s).

Check:
\[ e_{xx} = e_{11} = \frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x_1} = 0 \]
\[ e_{yy} = e_{22} = \frac{\partial v}{\partial y} = \frac{\partial u_2}{\partial x_2} = 0 \]
\[ e_{zz} = e_{33} = \frac{\partial w}{\partial z} = \frac{\partial u_3}{\partial x_3} = 0 \]
\[ e_{xy} = e_{12} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \frac{1}{2} (-C + C) = 0 \]
\[ e_{xz} = e_{13} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) = \frac{1}{2} (B - B) = 0 \]
\[ e_{yz} = e_{23} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) = \frac{1}{2} (-A + A) = 0 \]

Since \( e = 0 \) from above, there is no deformation associated with this displacement field.

Check now \( \mathbf{w} \), the rotation tensor:
\[ \omega_{11} = \omega_{22} = \omega_{33} = 0 \]
\[ \omega_{21} = -\omega_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right) = \frac{1}{2} (-C - C) = -C \]
\[ \omega_{31} = -\omega_{13} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) = \frac{1}{2} (B - (-B)) = B \]
\[ \omega_{32} = -\omega_{23} = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \right) = \frac{1}{2} (-A - A) = -A \]

Since the components of the rotation tensor are not all zeroes, the displacements of points in the body can be due to rotation (i.e. rigid-body motion) but not deformation (i.e. strain).
Example 4 on strains:
A 45° strain-rosette measures longitudinal strain along the axes shown in the figure. At a point P, the following measurements were made:

\[ e_{11} = 5 \times 10^{-4}, \ e_{1}^{'} = e_{1}^{''} = 4 \times 10^{-4}, \ e_{22} = 7 \times 10^{-4} \]

Determine the shear strain \( e_{12} \) at the point.

Solution:
Using strain tensor transformation, we can relate \( e_{11}' \) to \( e_{11}, \ e_{22}, \) and \( e_{12} \). Using such a relation, we can solve for the only unknown \( e_{12} \) as follows:

\[ e_{12}' = \beta_{11} e_{11}' + \beta_{12} e_{12}' + \beta_{13} e_{33}' \]

Where

\[
\beta = \begin{pmatrix}
\cos 45^\circ & \sin 45^\circ & 0 \\
-\sin 45^\circ & \cos 45^\circ & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[ \Rightarrow e_{1}' = e_{11}' + e_{12}' + e_{13}' \]

\[ \Rightarrow \beta_{11} e_{11}' + \beta_{12} e_{12}' + \beta_{13} e_{33}' \]

Velocity Fields in a Fluid and the Compatibility Condition:
For a fluid, we are more interested in the rate of displacement (i.e. velocity) of a fluid particle measured with respect to some coordinate system than simply with the displacement itself. This is because things are happening at a higher time rate in fluids undergoing deformation (typically called flow) than in solids.

Hence, instead of describing a displacement field for the continuum fluid material, once usually talks about a continuous velocity field with three components of velocity \( u, v \) and
$w$ which are functions of the spatial coordinates

$u = u(x_1, x_2, x_3)$, $v = v(x_1, x_2, x_3)$, $w = w(x_1, x_2, x_3)$

In index notation we simply write the above functions as

$v_i = v_i(x_1, x_2, x_3)$

Now considering two fluid particles $P$ and $P'$ which are located at some time next to one another at $x$ and $x + dx$ respectively. The difference in their velocities can be stated as

$$dv_i = \frac{\partial v_i}{\partial x_j} dx_j$$

Where the partial derivatives $\partial v_i / \partial x_j$ are evaluated at the particle $P$.

We can re-write the last equation as

$$\frac{dv_i}{dx_j} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)$$

$V_{ij}$ $\Omega_{ji}$

where $V_{ij}$ is called the “rate-of-deformation tensor” and $\Omega_{ji}$ is called the “spin tensor”.

Notice that $V_{ij} = V_{ji}$, i.e. it is symmetric, and that $\Omega_{ij} = -\Omega_{ji}$, i.e. it is anti-symmetric. As we learned before, for any anti-symmetric tensor one can always find a dual vector $\Omega$ that is related to this tensor

$\Omega_k = \varepsilon_{kij} \Omega_{ij}$ or $\Omega = \text{curl } v$

where the vector $\Omega$ is called the “vorticity vector”

Finally, similar to strains, the components of the rate-of-deformation tensor are coupled via an equation of compatibility (also called an “equation of integrability”) that is given in 2D by

$$\frac{\partial^2 V_{xx}}{\partial y^2} + \frac{\partial^2 V_{yy}}{\partial x^2} = 2 \frac{\partial^2 V_{xy}}{\partial x \partial y}$$