Chapter 10: Field Equations

Definition:

Previously, we have discussed in some details the static equilibrium equations:

\[
\frac{\partial \sigma_{ij}}{\partial x_j} + B_i = 0
\]

and the equations of motion:

\[
\frac{\partial \sigma_{ii}}{\partial x_j} + B_i = \rho \alpha_i = \rho \frac{Dv_i}{Dt}
\]

These partial differential equations describe the distribution of field quantities, in this case stress, in accordance with body forces and applied boundary conditions on displacements or velocities of points in the material. These two are an example of “Field Equations” which are essentially partial differential equations that govern the distribution of field quantities (stress, temperature, pressure in a fluid, etc.) subject to some boundary conditions on the continuum.

The Equation of Continuity (Conservation of Mass):

Consider a mass \( m \) enclosed in a domain \( V \) at a time \( t \). The mass would be given by:

\[
m = \int_V \rho \, dV
\]

, where \( \rho = \rho(x,t) \) is the density of the continuum at location \( x \) at time \( t \). Conservation of mass dictates that

\[
\dot{m} = \frac{Dm}{Dt} = \frac{dm}{dt} = 0
\]

where it is common to use capital \( D \) instead of small \( d \) in the derivative expression. This derivative we are typically used to in previous study is more appropriately termed the “material derivative”.

However, the total time derivative of any quantity, call it \( F \) where \( F = F(x,t) \), can be expressed using two partial derivatives using what is so called the “Spatial Description” as follows:

\[
\dot{F} = \frac{DF(x,t)}{Dt} = \left( \frac{\partial F}{\partial t} \right)_{x=\text{const}} + \nu_1 \frac{\partial F}{\partial x_1} + \nu_2 \frac{\partial F}{\partial x_2} + \nu_3 \frac{\partial F}{\partial x_3} = \frac{\partial F}{\partial t} + \nu_j \frac{\partial F}{\partial x_j} = \frac{\partial F}{\partial t} + \mathbf{v} \cdot \mathbf{\nabla} F
\]

The function \( F \) above can be any function that depends on spatial coordinates and time. A typical such function is the velocity field in a flowing liquid. Here, any point in the examination window of the flowing field will have a position \( (x_1, x_2, x_3) \) and at any given instant in time there will be a fluid particle instantaneously occupying that point in space and having some velocity vector associated with it, that is \( v_i(x_1, x_2, x_3, t) \) in component form. Note that the dependence on \( t \) indicates that for any other instant there might be another fluid particle in that \( (x_1, x_2, x_3) \) location that might have a different velocity.

More generally, \( F \) does not have to be only the velocity of a particle at location \( (x_1, x_2, x_3) \) and time \( t \), it can be any other property of that particle (temperature or density, for example).
Remember that for a continuum that is deforming or flowing smoothly, a current spatial location, i.e. \( x_i \), at any given time \( t \) will refer to a particle that was originally at \((a_1, a_2, a_3)\) when \( t = t_0 \) (i.e. at initial time):

\[
x_i = x_i(a_1, a_2, a_3, t), \quad i = 1, 2, 3
\]

Since the deformation/flow is smooth, the functions \( x_i = x_i(a_1, a_2, a_3, t) \) must be then continuous and differentiable functions. One can also find their inverse as long as they are one-to-one as such:

\[
a_i = a_i(x_1, x_2, x_3, t), \quad i = 1, 2, 3
\]

The material derivative of \( F \) means the time rate of change of the property \( F \) of the particle \((a_1, a_2, a_3)\) or formally

\[
\dot{F} = \frac{DF(x,t)}{Dt} = \left( \frac{\partial F(a_1, a_2, a_3, t)}{\partial t} \right)_{a}
\]

Since \( F(x_1, x_2, x_3, t) \) is then an implicit function of \( a_1, a_2, a_3 \) and \( t \), one can instead write

\[
\dot{F} = \frac{DF(x,t)}{Dt} = \frac{\partial F}{\partial t} \bigg|_x + \frac{\partial F}{\partial x_1} \frac{\partial x_1}{\partial t} \bigg|_a + \frac{\partial F}{\partial x_2} \frac{\partial x_2}{\partial t} \bigg|_a + \frac{\partial F}{\partial x_3} \frac{\partial x_3}{\partial t} \bigg|_a
\]

Now, knowing that

\[
v_i(a_1, a_2, a_3, t) = \frac{\partial x_i}{\partial t} \bigg|_{a=(a_1, a_2, a_3)}
\]

and

\[
\alpha_i(a_1, a_2, a_3, t) = \frac{\partial v_i}{\partial t} \bigg|_{a=(a_1, a_2, a_3)} = \frac{\partial^2 x_i}{\partial t^2} \bigg|_{a=(a_1, a_2, a_3)}
\]

which are the definitions of the \( i^{th} \) velocity and acceleration components of a particle originally at \((a_1, a_2, a_3)\). In light of our recent discussion above, these are also called the “material description” of velocity and acceleration. Lastly, substituting these definitions in the last equation for \( \dot{F} \) produces the first definition provided above for \( \dot{F} \).

Going back to our original problem now, which is to show that

\[
\dot{m} = Dm / Dt = dm / dt = 0
\]

and without proof, we can write

\[
Dm / Dt = \int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho v_j n_j dS = 0
\]

where \( S \) is the surface of the spatial volume \( V \), and \( n_j \) is the \( j^{th} \) component of the normal unit vector \( \mathbf{n} \) to the surface at the spot \( dS \). Using Gauss’s Theorem (also known as Green’s Theorem in the United States), the last surface integral can be converted into a
volume integral as follows:

\[
Dm / Dt = \int_{V} \frac{\partial \rho}{\partial t} dV + \int_{\nu} \frac{\partial}{\partial x_j} (\rho v_j) dV = \int_{\nu} \left( \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} \right) dV = 0
\]

\[
\Rightarrow \int_{\nu} \left( \frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial v_j}{\partial x_j} \right) dV = 0
\]

\[
\Rightarrow \int_{\nu} \left( \frac{D \rho}{D t} + \rho \frac{\partial v_j}{\partial x_j} \right) dV = 0
\]

Since the above integral equations must hold for any arbitrary domain \( V \), the integrand of these equations must then be equal to zero:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0
\]

\[
\frac{D \rho}{D t} + \rho \frac{\partial v_j}{\partial x_j} = 0
\]

These equations above are called the “equations of continuity” which assures mass conservation.

In problems involving statics only and not dynamics, the above equations are satisfied identically since there is no dependence on time nor there is motion occurring.

### The Navier-Stokes Equations

The stress-strain rate relationship for an isotropic fluid is given by:

\[
\sigma_{ij} = -p \delta_{ij} + \lambda V_{kk} \delta_{ij} + 2 \mu V_{ij}
\]

which can be rewritten as

\[
\sigma_{ij} = -p \delta_{ij} + \lambda \frac{\partial v_k}{\partial x_k} \delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)
\]

i.e.

\[
\sigma_{xx} = -p + 2 \mu \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)
\]

\[
\sigma_{yy} = -p + 2 \mu \frac{\partial v}{\partial y} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)
\]

\[
\sigma_{zz} = -p + 2 \mu \frac{\partial w}{\partial z} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)
\]

\[
\sigma_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad \sigma_{xz} = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \quad \sigma_{yz} = \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)
\]
Now substituting the equations of motion into the last equations, one obtains what is so called the “Navier-Stokes equations”:

$$\rho \frac{Dv_i}{Dt} = \rho X_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i} \left( \lambda \frac{\partial v_k}{\partial x_k} \right) + \frac{\partial}{\partial x_i} \left( \mu \frac{\partial v_k}{\partial x_k} \right) + \frac{\partial}{\partial x_k} \left( \mu \frac{\partial v_i}{\partial x_k} \right)$$

where $X_i$ stands for the body force per unit mass.

The continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_k)}{\partial x_k} = 0$$

must also be satisfied for the fluid flow.

Assuming an incompressible fluid, we get

$$\rho = \text{const.}$$

In this case, the equation of continuity reduces to:

$$\frac{\partial v_k}{\partial x_k} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

and the Navier-Stokes equations reduce to:

$$\rho \frac{Dv_i}{Dt} = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_k \partial x_k}$$

which, in long-hand, can be written as:

$$\frac{Du}{Dt} = X - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \nabla^2 u,$$

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \nabla^2 v,$$

$$\frac{Dw}{Dt} = Z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\mu}{\rho} \nabla^2 w$$

where the ratio $\mu/\rho$, sometimes termed “$\nu$”, is called the “kinematic viscosity”. The last three equations along with the equation of continuity for an incompressible viscous fluid comprise four equations in four unknowns ($u, v, w$ and $p$) which can, in principle, be solved subject to certain boundary conditions on the flow.

Example: Laminar Flow in a Horizontal Channel

Consider steady flow (i.e. flow that does not change with time, or depend on time in other words) of an incompressible fluid through a horizontal channel of width $2h$ between two parallel fixed planes/plates as shown in the figure (Note: “laminar” flow is one in which the flow is essentially proceeding through the glide of lamellas, or layers, of fluid on top of one another and therefore not much or no vigorous mixing of fluid particles is taking place which would then be called “turbulent” flow. An example of laminar flow is when you open the water faucet slowly and lightly, and an example of turbulent flow is when you have it wide open and the water is flowing out under a lot of pressure in the pipes).
Figure: Laminar flow in a 2D channel showing a cross-section of the flow profile which repeats itself uniformly all along the x-axis.

Solution:
From the shape or profile of the fluid flow, it can be seen that
\[ u = u(y), \quad v = 0, \quad w = 0 \quad (*) \]
Hence, we need to find the function \( u(y) \) to solve the problem.

Whatever is such function, it has to satisfy the Navier-Stokes equations and the equation of continuity plus any boundary conditions that might exist.

The boundary conditions in this problem are:
\[ u(y = +h) = 0, \quad u(y = -h) = 0 \]
Since the top and bottom surfaces are no-slip boundaries meaning that any fluid particles at those boundaries adhere or attach to the surface, i.e. they wet it, and hence they do not have any relative motion with respect to these boundaries. Since the boundaries are not moving, i.e. they are static, hence the boundary condition above.

First, note that the above equations (*) identically satisfy the continuity equation
\[
\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial v_j}{\partial x_j} = 0
\]
Also, the Navier-Stokes equations in this case, assuming no body forces, which is a good assumption since no magnetic or electrostatic forces operate and the observation that there is no motion in the y-direction which is the line of action of gravitational forces, reduce to
\[ 0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2}, \]

\[ 0 = \frac{\partial p}{\partial y}, \]

\[ 0 = \frac{\partial p}{\partial z}. \]

The last equations indicate that the pressure \( p \) is a function of \( x \) only.

Moreover, since \( \frac{d^2 u}{dy^2} \) is a function of \( y \) only and \( \frac{\partial p}{\partial x} \) is a function of \( x \) only, then we must have both \( \frac{d^2 u}{dy^2} \) and \( \frac{\partial p}{\partial x} \) each equal to some constant according to

\[ 0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2}. \]

Assume this constant for \( \frac{\partial p}{\partial x} \) to be \(-\alpha\). The last equation then becomes

\[ \frac{d^2 u}{dy^2} = -\frac{\alpha}{\mu} \]

which can be solved to be

\[ u = A + By - \frac{\alpha y^2}{\mu} \frac{2}{2} \]

The constants of integration \( A \) and \( B \) can be determined from the boundary conditions to yield the final solution

\[ u = \frac{\alpha}{2\mu} (h^2 - y^2) \]

Thus the velocity profile is parabolic with maximum velocity occurring at the center of the conduit/channel, i.e. at \( y = 0 \), and equal to \( u = \frac{\alpha}{2\mu} h^2 \). Note that the velocity is linearly proportional in this case the pressure gradient \( \frac{\partial p}{\partial x} \) which is said to “drive the flow”.