

Shallow free-surface flows.

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Abstract

The analysis of thin film flows with large distortions of the free surface is discussed. The work done by various research groups is presented and compared. The investigation done for a circular hydraulic jump and the gravity driven flow down an inclined plane leads to a simple stationary model consisting of two differential equations. The results compare well with experiments done at moderate Reynolds numbers. Analysis of flows with additional eddies at high Reynolds numbers yields interesting results which serve as a basis for future work.

1 Introduction

This paper explains the quantitative methods to describe free surfaces shallow flows with separation. Two cases are considered: the circular hydraulic jump and the flow down an inclined plane. The method used by Watanabe et al. [1] is capable of handling internal eddy and separated flow. They use a variable profile as in the Karman-Polhausen approximation which gives a system of two differential equations for stationary states that can smoothly go through the jump. For low Reynolds numbers the lubrication approximation and the asymptotic wave theory can be effectively used but it is not so for high Reynolds numbers. Watanabe et al. [1] developed the theory for the hydraulic jump by studying the boundary layer approximation to the full Navier-Stokes equations and reduce it to a simple set of equations by averaging over the thickness. For the flow down an inclined plane the authors seek a stationary solution approaching equilibrium Nusselt flow far downstream. It is demonstrated that solutions can be found only by introducing flexibility in the velocity profile and no solution can be found for a self-similar velocity profile. The velocity profile departs considerably from parabolic near the jump. It is found difficult to analyze the stability of the solutions with jumps. In inviscid theory, the flows are classified as super- and subcritical when the thickness is small and large, respectively. The authors classify a stationary flow into these

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two categories but their model shows spurious divergences in the short-wavelength region , and this makes it suitable only for stationary solutions. However the model is capable of handling kink-like traveling wave solutions studied by other authors. The assumptions were stated by Bohr et al. (1997) [4]. Work done by Ruschak et. al. (2001) [11] was similar where they analyzed the flow down an inclined plane. They compared the predictions from the model with experiments with full numerics of the Navier-Stokes and the boundary layer equations. Bohr et al. (1993) [2] studied the linear stability of the equilibrium flow.

2 Theory

2.1 Circular hydraulic jump

A most common occurrence even observed in the kitchen sink is when a jet of fluid emerging from the tap hits the sink bowl and spreads out radially in a thin, flowing layer. At a distance from the jet a sudden thickening of the flow takes place which researchers have been studying what is called the circular hydraulic jump. This occurrence in the kitchen sink takes place at very high Reynolds numbers and disturbances make the jump highly non stationary and distorted. However in controlled laboratory experiments a moderate Reynolds number can be realized and the flow is laminar. Many researchers like Tani et al. [12], Ihigai et al. [8] have studied the phenomenon experimentally at moderate Reynolds numbers and the flows were laminar. These assumptions were also made by Watanabe et al. [1] in their integral analysis.

In experimental analysis the jump was formed on a flat disc with a circular rim. The rim height d can be varied, and it is an important control parameter. The rim is located far from the impinging jet so that it does not affect the jump except that it changes the height of the fluid layer h_{ext} exterior to the jump. A picture and schematic are shown in the following figure.

The experiments show that the jump forms even when $d = 0$ but a larger d makes h_{ext} larger and this results in a stronger jump. Bohr et al. (1998) [13] reported that as the jump remains stable when d is small however as d increases a wave breaking transition occurs which results in an additional eddy called a roller or surfing wave. This was also observed by Liu and Lienhard (1993) [10] but the Reynolds number was too large to show the transition. This resembles a broken wave in the ocean but still it is laminar. This transition is often observed to break symmetry. Ellegaard et al. (1999) [6] observed that an interesting set of polygonal jumps is created rather than a circular one. Watanabe [1] and others have however concentrated on the case with no transition. They report that the approach considered standard for analyzing hydraulic jumps is to combine the inviscid shallow water equation with Rayleighs shocks. This was also reported by Chow (1959) [5].

In a coordinate system that moves with the shock, the flow velocity v_1 and height h_1 upstream of the jump as well as v_2 and h_2 downstream of the jump are taken to be positive values. Then, the conservation of mass flux Q across the jump is given by $v_1 h_1 = v_2 h_2 = Q$. Conservation of momentum flux is $h_1(v_1^2 + gh_1/2) = h_2(v_2^2 + gh_2/2)$. These shock conditions lead to the relation.

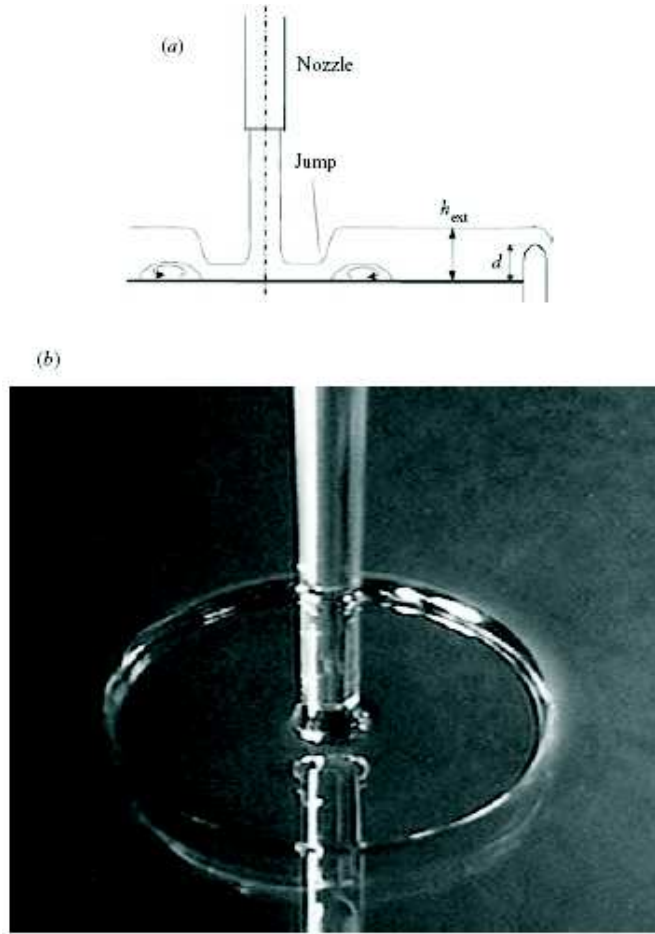


Figure 1: (a) Schematic view of the circular hydraulic jump. (b) Snapshot of a nearly perfect stationary and circular hydraulic jump. Ethylene-glycol is the liquid. (after Watanabe *et al.* 2002)

$$\frac{h_2}{h_1} = \left[-1 + [1 + 8F_1^2]^{0.5} \right] / 2 = 2 / \left[-1 + [1 + 8F_2^2]^{0.5} \right] \quad (1)$$

Where F_1 and F_2 are the upstream and downstream Froude numbers and h_c is called the critical height. The jump connects a super critical flow with $F > 1$ on the shallower side ($h < h_c$) to a subcritical flow on the deeper side with ($h > h_c$). The Froude number measures the ratio of fluid velocity to the velocity of the linear surface waves, it means that the fluid moves more rapidly than the linear surface waves on the shallower side, but moves slower on the deeper side.

Watanabe *et al.* [1] have applied this theory combined with an assumption of potential flow, to describe the circular hydraulic jump however they have found that this leads to an incorrect estimate of the radius of the jump r_{jump} . They observed that treating the jump as a discontinuity provides no information with the internal structure of the jump region. They observed that in the presence of viscous loss, the assumption that assumes a Rayleigh shock seems inconsistent. They found it plausible to attribute the energy dissipation entirely to

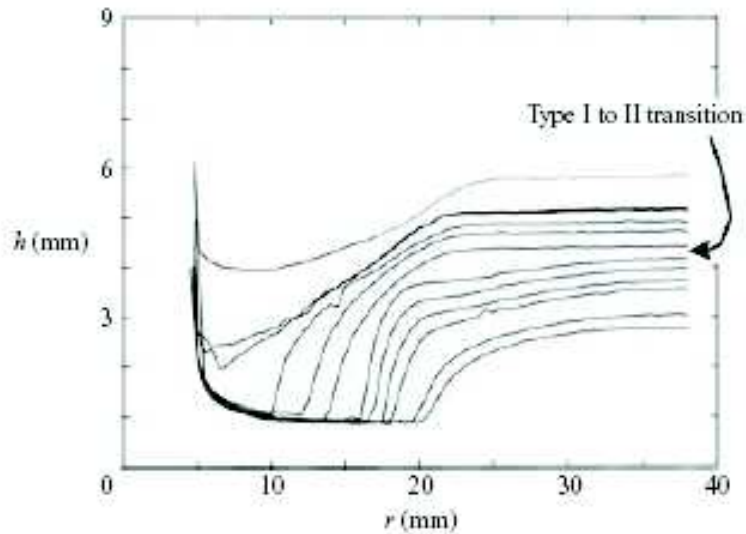


Figure 2: Height profiles $h(r)$ for different values of the external height h_{ext} . The height $h(r)$ approaches h_{ext} for large values of r . (after Watanabe et al. 1996)

laminar viscous forces and to construct a viscous theory which provides a smooth but kink-like surface shape without the need for discontinuity.

A similar approach was used by Higuera (1997) [7] where they found that, for the free-stream flow above the free surface it is possible to modify the boundary layer iteratively by taking the pressure gradient across the layer into account. The height of the free surface enters the equations through the hydrostatic pressure. Watanabe [1] and his group have proposed to resolve the problem in order to obtain a simpler system that describes the coupling mechanism. They included an additional degree of freedom in the velocity profile to make it non-self-similar, as in the Karman-Polhausen method for the usual boundary layer theory. The resulting model for a stationary solution is two coupled ordinary differential equations and reproduces a flow without transition. The stationary solution in the model can be analyzed analytically. They introduced a formal parameter β where $\beta = 1$ corresponds to flow without transition whereas Rayleigh's shock condition is realized in the limit $\beta = 0$. They also described the full model of the circular hydraulic jump under the assumption that the flow is laminar and radially symmetric without any angular velocity component. However they proposed that analyzing the full model is a formidable task and some simplification needs to be made. The Reynolds number for the flow of the circular hydraulic jump is too large to satisfy the lubrication approximation but not large enough to use the inviscid approximation. They used a method of truncation of the full model by the boundary layer approximation which had the boundary layer approximation pressure, viscous and inertial terms. From this they derived the stationary boundary layer equation:

$$uu_r + wu_z = -h + u_{zz} \quad (2)$$

where the prime denotes the derivative with respect to r . Rather than solving the partial

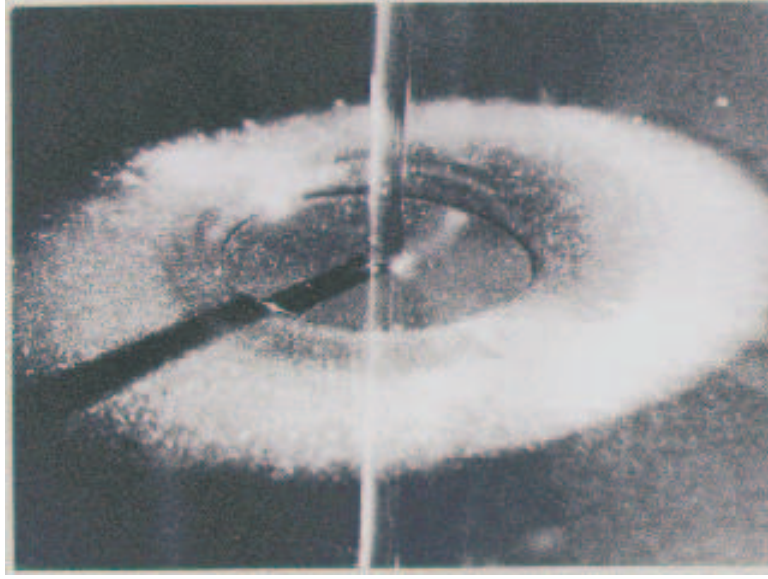


Figure 3: Snapshot of a circular hydraulic jump demonstrating the transition between the two types of motion that takes place in a surprisingly abrupt manner, i.e. in a narrow annular region, defining rather precisely a radius, where the hydraulic jump takes place. (after Haaning 1996).

differential equation they satisfied the mass and momentum conservation laws, derived from averaging over the transverse z -direction. For this they made an ansatz for the radial velocity profile u . They showed that the model is capable of going through the jump smoothly once enough flexibility is introduced in the assumed profile. They assumed a simple ansatz for a self-similar velocity profile:

$$\frac{u[r, z]}{v[r]} = f[\eta] \quad (3)$$

The other ansatz they investigated was more flexible and was used for resolving the flow pattern in the vicinity of the jump. This follows the ideas developed by von Karman and Polhausen for the usual boundary layer flow around a body. There, separation of the boundary layer can occur when the pressure gradient, imposed by the external inviscid flow, becomes adverse. In the analysis done by Watanabe et al. [1] there was no external flow, but there was a pressure gradient, along the bottom $z = 0$, that is proportional to $h(r)$ due to the hydrostatic pressure. Thus, the possibility arises that the flow separates on $z = 0$ near the jump where $h(r)$ is large and pressure is increasing in r , as in the usual boundary layer flow.

To improve the parabolic profile they approximate the velocity profile by a cubic equation:

$$u(r, z)/v(r) = a\eta + b\eta^2 + c\eta^3 \quad (4)$$

where a, b, c are functions of r . Using the averaged momentum equation and the Karman Polhausen method they obtained a non-autonomous system of two ordinary differential equations:

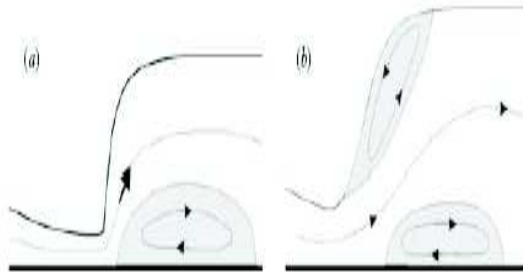


Figure 4: A schematic picture showing two observed flow patterns: (a) Flow, with a separation bubble, which occurs for small d , and (b) Flow, with an additional roller eddy, for large d . Transitions between these states occur at a certain d , with a small hysteresis. (after Watanabe et al. 2002)

$$h^1 = -\frac{5\lambda + 3}{rh^3} \quad (5)$$

$$\frac{dG}{d\lambda} \lambda^1 = \frac{4r\lambda}{h} + G(\lambda) \frac{[h^4 - (5\lambda + 3)]}{rh^4} \quad (6)$$

This was the model for the stationary circular hydraulic jump. This simplified model did contain solutions which describe the observed jumps. This model was derived ignoring short wavelengths and surface tension but it provided a good description of the regions before the jump, after the jump and the jump itself. Similar approach was found in the works of other research groups with convincing continuous solutions through the jump.

2.1.1 Numerical Analysis

Watanabe et al. [1] also demonstrated a numerical simulation of this integral model consisting of two ordinary differential equations. They solved the model as a boundary value problem by specifying two boundary conditions for different values of r . The values are taken from the measured surface height data. There is no fitting parameter once they are chosen, and the function $h(r)$ and the shape parameter $\lambda(r)$ are determined. In particular, the shape parameter need not be specified as a part of the boundary conditions. This is an advantage of the simplified model since one no longer needs to specify the velocity profile at the inlet and/or outlet boundaries, which is not easy to do. In fact specifying both h and λ at one r , either inside the jump or outside, and solving as an initial value problem is unstable. The system is extremely sensitive to the initial condition if one integrates the differential equations in the direction of increasing r from a small r or in the direction of decreasing r from a large r . Therefore, r_1 and r_2 are chosen close to 1. Then, a straightforward shooting method from either boundary is sufficient to obtain a solution. After this is achieved, the solution is extended backwards from r_1 and forwards from r_2 respectively. They found integrations in these directions were found to be stable.

2.2 Flow down an inclined lane

Watanabe et al. [1] describe the applicability of the method of analysis used for cylindrical hydraulic jump in two dimensional Cartesian geometry. They considered a fluid stream running down an inclined plane under the influence of gravity. They followed the same strategy to derive the averaged partial equations. They also compare models based on a self-similar and one-parameter velocity profiles.

Gravity driven flows down an inclined based have also been studied by various research groups with the pioneering work done by Kapitsa and Kapitsa (1949) [9] where they dealt with the time evolution of surface waves. Watanabe [1] and others assume stationarity in their analysis of the partial differential equations due to which they reduce to a system of ordinary differential equations. There is a unique equilibrium flow, corresponding to a fixed point in the stationary equations. It is, thus, impossible to find a solution with a jump which connects two equilibrium flows, as in Rayleighs shock. In their viscous model they demonstrate that a solution with a jump which connects a transient flow to an equilibrium flow is possible if a shape parameter is included.. Their analysis also provides information on the limitations of the one-parameter model and on the criterion for super- or subcriticality of the flow. The one-parameter model is unsuited to time-dependent calculations of chaotic wave trains for instance, but is capable of finding solitary traveling fronts that could occur when the influx of fluid upstream is suddenly changed. It turns out, however, that these fronts are not analogous to the circular hydraulic jump in the sense that the velocity profiles deviate little from parabolic. In addition, the definition of super- or subcriticality is with respect to the front, and its criterion becomes trivial.

2.2.1 Governing equations

It is observed that the averaged momentum equation depends on the choice of the velocity profile. For a self-similar velocity profile a governing equation is as follows,

$$[hv_t] + G [hv^2]_x = 3h/R - 3hh_x/[R \tan \alpha] - 3v/[Rh] + Whh_{xxx} \quad (7)$$

,where G is a constant for the chosen profile and R and W are Reynolds and Weber numbers respectively, suitable for this geometry.

For a one-parameter velocity profile with a third order polynomial the governing equations are:

$$[hv]_t + [Hv^2G]_x = 4v\lambda/[Rh] \quad (8)$$

$$h_x \cot \alpha = 1 - v[5\lambda + 3]/[3h^2] + WRH_{xxx}/3 \quad (9)$$

Rushack et al. [11] tested a one-parameter family of conditions for the above equation. They found that evaluation on the bottom yielded the best quantitative agreement with their extensive computations of the full Navier-Stokes and boundary layer equations.

Watanabe et al. [1] have described solutions for the similarity model and the one-parameter model that propogate down the plane with a constant velocity $c > 0$. Travelling waves are descibed by this model. The one-parameter model is proposed to have a different

family of stationary solutions. The two models are reduced to ordinary differential equations that are treated as dynamical systems with the moving distance x as the independent variable. They found that the presence of surface tension makes the order of the equations higher and finding trajectories is a then a difficult task. Any trajectories of the reduced dynamical systems correspond to stationary flows, but the most interesting ones are those convergent to the fixed point (the Nusselt flow) sufficiently far downstream.

The similarity model reduces to a single equation,

$$\frac{dh}{dx} = [h - 1] [h^2 + h + 1] / h^3 / \tan \alpha - 2R/5 \quad (10)$$

Due to the first order nature of this equation, trajectories converge to the equilibrium point monotonically. The one parameter model results in,

$$[h^{-1}G]_x = 4\lambda / [Rh^2] \quad (11)$$

$$h_x \cot \alpha = 1 - [5\lambda + 3] / 3h^3 \quad (12)$$

This is a two-dimensional system, but the trajectory convergent to the fixed point is computed as its stable manifold.

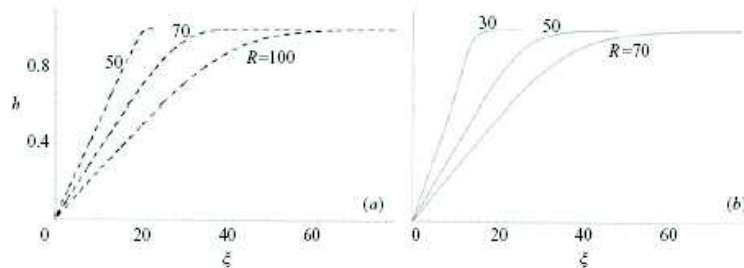


Figure 5: Computed stationary solutions for $\alpha = 3$ degrees and $c = 0$. Dashed curves are solutions for the similarity model. Solid curves are solutions of the one-parameter model. A larger R corresponds to a slower convergence to equilibrium for $h = 1$. These solutions do not show any shock like structure. (after Watanabe et al. 2002)

It can be seen in Figure 5 that the trajectories are monotonically approaching the equilibrium height if R is sufficiently large. They are qualitatively identical to the ones from the similarity model. However, for smaller R , a different type of trajectory is obtained as shown in Figure 5. Both the height profile and the shape parameter vary rapidly at a certain distance, creating a jump structure. This is the two-dimensional version of the jump that Watanabe et al. [1] obtained. This type of solution however does not connect two equilibrium flows, but an upstream transient flow to a downstream equilibrium one. It could be realized, for instance, as a stationary flow exiting a sluice gate.

3 Conclusion

Watanabe et al. [1] have demonstrated a simple but fairly quantitative method of reducing flows with strongly deformed free surfaces to a manageable system of equations. By assuming a 'flexible' velocity profile whose shape parameter is another dependent variable they have described flows with an internal eddy. In radial geometry their results compare well with the experiments.

They have used a similar model to describe flow down an inclined plane. Introducing a shape parameter is found to be crucial. They have constructed a descent model for small Reynolds numbers where a flexible internal velocity profile can be adopted and the analysis is in good agreement with the controlled laboratory experiments at moderate Reynolds numbers.

Roberts (1996) [14] used the centre manifold theory and obtained a two-variable model which captures qualitatively new effects absent in the similarity model with the parabolic profile.

Sobey (2000) [15] described the jump region more accurately using the application of triple-deck theory. The method described by Watanabe et al. [1] however fails to predict the transition in the circular jump. They quote that a more elaborate treatment of the terms neglected in the boundary layer approximation is needed. As stated, their future work would include a more detailed analysis along with consideration of the flow outside the jump region and the derivation of an approximation of the surface profile.

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