IV. Compressible flow of inviscid fluids

- Governing equations for $\nu = 0$, $\rho \neq const$:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0$$

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{u} = -\boldsymbol{\nabla} p$$

$$\rho \frac{De}{Dt} = \rho \frac{\partial e}{\partial t} + \rho (\boldsymbol{u} \cdot \boldsymbol{\nabla}) e = -p \boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T)$$

$$p = p(\rho, T), \quad e = e(\rho, T)$$

Alternate forms of energy equation

Equation for enthalpy $h = e + p/\rho$

$$\rho \frac{\partial h}{\partial t} + \rho \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} \right) h = \frac{\partial p}{\partial t} + \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} \right) p + \boldsymbol{\nabla} \cdot \left(k \boldsymbol{\nabla} T \right)$$

(derivation – similar to what we did in Ch. 10)

Equation for negligible heat conduction and perfect gas

Start with

$$\rho \frac{De}{Dt} = -p \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

For perfect ideal gas,

$$e = e(T) = c_v T, \quad p = \rho R T$$

In this case

$$\frac{De}{Dt} = \frac{De}{DT}\frac{DT}{Dt} = c_v \frac{DT}{Dt}$$

From continuity equation,

$$\nabla \cdot (\rho \boldsymbol{u}) = \nabla \rho \cdot \boldsymbol{u} + \rho \nabla \cdot \boldsymbol{u} = -\frac{\partial \rho}{\partial t}$$

$$\rho \nabla \cdot \boldsymbol{u} = -\left(\frac{\partial \rho}{\partial t} + (\boldsymbol{u} \cdot \nabla)\rho\right) = -\frac{D\rho}{Dt}$$

Insert

$$\frac{De}{Dt} = c_v \frac{DT}{Dt} \qquad \text{and} \qquad \nabla \cdot \boldsymbol{u} = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

Into the energy equation -

$$c_v \frac{DT}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt}$$

Recall that (for ideal gas) $T = p/(\rho R)$

$$c_v \frac{D}{Dt}\left(\frac{p}{\rho R}\right) = \frac{p}{\rho}\frac{D\rho}{Dt}$$

$$\frac{c_v}{R}\frac{1}{\rho^2}\left(\rho \frac{Dp}{Dt} - p\frac{D\rho}{Dt}\right) = \frac{p}{\rho}\frac{D\rho}{Dt}$$

$$\frac{c_v}{R}\frac{Dp}{Dt}=\frac{c_v}{R}\frac{p}{\rho}\frac{D\rho}{Dt}+\frac{p}{\rho}\frac{D\rho}{Dt}$$

Multiply both parts by $R/(c_v p)$

$$\frac{1}{p}\frac{Dp}{Dt}=\frac{1}{\rho}\frac{D\rho}{Dt}+\frac{R}{c_v}\frac{1}{\rho}\frac{D\rho}{Dt}=\frac{1}{\rho}\frac{D\rho}{Dt}\left(1+\frac{R}{c_v}\right)$$

Note that $R = c_p - c_v$, so

$$1+\frac{R}{c_v}=\frac{c_v+c_p-c_v}{c_v}=\frac{c_p}{c_v}=\gamma$$

With that,

$$\frac{1}{p}\frac{Dp}{Dt}=\frac{\gamma}{\rho}\frac{D\rho}{Dt}, \text{ or } \qquad \frac{D}{Dt}\left(\log p-\log\rho^{\gamma}\right)=0$$

Integrate for the same fluid volume:

$$\frac{p}{\rho^{\gamma}} = const$$

Isentropic law (inviscid fluid, no heat transfer)

11.1. Propagation of infinitesimal disturbances

(a.k.a. linear acoustics)

- Assumptions
  - 1D setting (x-axis only)
  - Perfect gas initially at rest
  - No heat conduction
  - Small disturbance propagates in x-direction
- Governing equations
  - Continuity (1D):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \ \rightarrow \ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

- Momentum (1D)

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}=-\boldsymbol{\nabla}\,p \quad\rightarrow\quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

- Energy (perfect gas, no heat transfer)

$$\frac{p}{\rho^{\gamma}}=const$$

For isentropic gas, $p=p(\rho)$, so

$$\frac{\partial p}{\partial x}=\frac{dp}{d\rho}\frac{\partial \rho}{\partial x}$$

Rewrite continuity and momentum with this -

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0 \qquad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial \rho}{\partial x}=0$$

Assumptions for linearization

Hydrostatic value

$$p = p_0 + p'$$
$$\rho = \rho_0 + \rho'$$
$$u = u'$$

Assume perturbed (') values to be small, plug into governing equations...

Product: 2nd order

$$\frac{\partial}{\partial t}\left(\overset{\rho_0 = const}{\cancel{\rho_0}} + \rho'\right) + u'\frac{\partial}{\partial x}\left(\overset{\rho_0 = const}{\cancel{\rho_0}} + \rho'\right) + \left(\rho_0 + \rho'\right)\frac{\partial u'}{\partial x} = 0$$

Product: 2nd order

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0$$

Product: 2nd order

Not a function of x

$$\frac{\partial u'}{\partial t} + u'\frac{\partial u'}{\partial x} + \frac{1}{\rho}\frac{d\left(p_0 + p'\right)}{d\rho}\frac{\partial\left(\rho_0 + \rho'\right)}{\partial x} = 0$$

Product: 2nd order

Note that

$$\frac{d\left(p_0 + p'\right)}{d\rho} = \left.\frac{d\,p}{d\rho}\right|_{p=p_0,\rho=\rho_0} + p'\left.\frac{d^2 p}{d\rho^2}\right|_{p=p_0,\rho=\rho_0} + \ldots$$

$$\frac{1}{\rho} = \frac{1}{\rho_0 + \rho'} = \frac{1}{\rho_0}\frac{1}{1 + \rho'/\rho_0} = \frac{1}{\rho_0}\left(1 - \frac{\rho'}{\rho_0} + \ldots\right)$$

Thus after linearization

$$\frac{\partial u'}{\partial t} + \frac{1}{\rho_0}\left.\frac{d\,p}{d\rho}\right|_0\frac{\partial \rho'}{\partial x} = 0$$

Differentiate first equation (continuity) in t

$$\frac{\partial^2 \rho'}{\partial t^2} + \rho_0 \frac{\partial^2 u'}{\partial x \partial t} = 0$$

Take second equation (momentum), multiply by ρ_0, differentiate in x

$$\rho_0 \frac{\partial^2 u'}{\partial t \partial x} + \left.\frac{dp}{d\rho}\right|_0 \frac{\partial^2 \rho'}{\partial x^2} = 0$$

Subtract the two equations to eliminate the cross-differential term

$$\frac{\partial^2 \rho'}{\partial t^2} - \left.\frac{dp}{d\rho}\right|_0 \frac{\partial^2 \rho'}{\partial x^2} = 0$$

Now differentiate continuity in x

$$\frac{\partial^2 \rho'}{\partial t \, \partial x} + \rho_0 \frac{\partial^2 u'}{\partial x^2} = 0$$

...and momentum in t

$$\frac{\partial^2 u'}{\partial t^2} + \frac{1}{\rho_0} \left. \frac{d\,p}{d\,\rho} \right|_0 \frac{\partial^2 \rho'}{\partial x \, \partial t} = 0$$

Multiply first equation by...

$$\frac{1}{\rho_0} \left. \frac{d\,p}{d\,\rho} \right|_0$$

Then subtract first equation from the second

$$\frac{\partial^2 u'}{\partial t^2} - \left.\frac{d\,p}{d\rho}\right|_0 \frac{\partial^2 u'}{\partial x^2} = 0$$

Let

$$a_0 = \sqrt{\left.\frac{dp}{d\rho}\right|_0}$$ Speed of sound

Equations can be rewritten as

$$\rho'_{tt} - a_0^2 \rho'_{xx} = 0$$
$$u'_{tt} - a_0^2 u'_{xx} = 0$$

D'Alembert's equations (wave equations)

General solution for ρ' (same form for u')

$$\rho' = f\left(x - a_0 t\right) + g\left(x + a_0 t\right)$$

Wave traveling right

Wave traveling left

For this theory (linear acoustics) to work, must have:

$$\frac{\rho'}{\rho_0} \ll 1, \quad \frac{p'}{p_0} \ll 1, \quad \frac{u'}{a_0} \ll 1$$

Speed of sound in perfect gas – evaluate using isentropic energy equation

$$\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma}, \qquad p = \rho^\gamma \frac{p_0}{\rho_0^\gamma}$$

$$\frac{dp}{d\rho} = \gamma \rho^{\gamma-1} \frac{p_0}{\rho_0^\gamma} = \frac{\gamma}{\rho} \rho^\gamma \frac{p_0}{\rho_0^\gamma} = \frac{\gamma}{\rho} p$$

Also for ideal gas $p = \rho RT$, so

$$\frac{dp}{d\rho} = \frac{\gamma}{\rho} p = \gamma R T$$

Thus for the speed of sound we can write

$$a_0 = \sqrt{\gamma R T_0} = \sqrt{\gamma \frac{p_0}{\rho_0}}$$

11.2. Propagation of finite disturbances

Start with governing equations from previous section before linearization...

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial \rho}{\partial x}=0$$

$$\frac{p}{\rho^{\gamma}}=const$$

From previous section, $u=u(\rho), p=p(\rho)$ only (both also depend on initial conditions)

If $u = u(\rho)$, we can also rewrite that as $\rho = \rho(u)$

Assuming the same dependence for finite-amplitude case ($\rho = \rho(u)$, $p = p(\rho)$), use chain rule for derivatives in governing equations

$$\frac{\partial \rho}{\partial t} = \frac{d\rho}{du}\frac{\partial u}{\partial t} \qquad \frac{\partial \rho}{\partial x} = \frac{d\rho}{du}\frac{\partial u}{\partial x} \qquad \frac{\partial p}{\partial x} = \frac{dp}{d\rho}\frac{d\rho}{du}\frac{\partial u}{\partial x}$$

Continuity equation becomes...

$$\frac{d\rho}{du}\frac{\partial u}{\partial t} + u\frac{d\rho}{du}\frac{\partial u}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{d\rho}{du}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \rho\frac{\partial u}{\partial x} = 0$$

For the momentum equation...

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{dp}{d\rho}\frac{d\rho}{du}\frac{\partial u}{\partial x}=0$$

From continuity,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-1/\left(\frac{d\rho}{du}\right)\rho\frac{\partial u}{\partial x}=-\frac{du}{d\rho}\rho\frac{\partial u}{\partial x}$$

Plug that into momentum, move pressure term to the right, lose the - signs

$$\rho\frac{du}{d\rho}\frac{\partial u}{\partial x}=\frac{1}{\rho}\frac{dp}{d\rho}\frac{d\rho}{du}\frac{\partial u}{\partial x}$$

$$\rho\frac{du}{d\rho}=\frac{1}{\rho}\frac{dp}{d\rho}\frac{d\rho}{du}$$

$$\left(\frac{du}{d\rho}\right)^2 = \frac{1}{\rho^2}\frac{dp}{d\rho} \qquad\qquad \frac{du}{d\rho} = \pm\frac{1}{\rho}\sqrt{\frac{dp}{d\rho}}$$

Let

$$a = \sqrt{\frac{dp}{d\rho}}$$

For small amplitude, $a \to a_0$ (speed of sound)

Then

$$\frac{du}{d\rho} = \pm\frac{a}{\rho}, \qquad \frac{du}{a} = \pm\frac{d\rho}{\rho}$$

Use these expressions to alter the momentum equation...

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\overset{a^2}{\frac{dp}{d\rho}}\overset{\pm\frac{\rho}{a}}{\frac{d\rho}{du}}\frac{\partial u}{\partial x} = 0$$

For a forward-propagating wave ($\pm \to$ **+**),

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial x} = 0$$

Rewrite this as

$$\frac{\partial u}{\partial t} + \left(u + a\right)\frac{\partial u}{\partial x} = 0$$

A general solution of this equation (forward wave)

Any differentiable function (particular solution – from IC)

$$u = f\left(x - \left(u + a\right)t\right)$$

Functions of x,t

Use the polytropic equation to rewrite a in terms of speed of sound a_0

$$\frac{p}{\rho^{\gamma}} = \frac{p_0}{\rho_0^{\gamma}}$$

$$\frac{d\,p}{d\,\rho} = \frac{d}{d\,\rho}\left(\rho^{\gamma}\frac{p_0}{\rho_0^{\gamma}}\right) = \gamma\,\rho^{\gamma-1}\frac{p_0}{\rho_0^{\gamma}}$$

$$a = \sqrt{\frac{d\,p}{d\,\rho}} = \sqrt{\gamma\,\rho^{\gamma-1}}\left(\frac{p_0}{\rho_0^{\gamma}}\right)^{1/2} = \sqrt{\gamma\frac{p_0}{\rho_0}}\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}}$$

From previous section,

$$a_0 = \sqrt{\gamma\frac{p_0}{\rho_0}}$$

Thus

$$a = a_0 \left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{2}}$$

Recall that we had...

$$\frac{d\,u}{d\,\rho} = \pm \frac{a}{\rho}$$

For a forward wave (+),

$$d\,u = a \frac{d\,\rho}{\rho}$$

Rewrite this using the expression with a_0

$$d\,u = a_0 \left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{2}} \frac{d\,\rho}{\rho} = \frac{a_0}{\rho_0^{\frac{\gamma-1}{2}}} \rho^{\frac{\gamma-1}{2}-1} d\,\rho$$

$$du = \frac{a_0}{\rho_0^{(\gamma-1)/2}} \rho^{(\gamma-3)/2} \, d\rho$$

Integrate this in ρ to get

$$u = \frac{2}{\gamma-1} \frac{a_0}{\rho_0^{(\gamma-1)/2}} \rho^{(\gamma-1)/2} + const$$

When $\rho \to \rho_0$, $u \to 0$:

$$0 = \frac{2}{\gamma-1} a_0 + const$$

Thus

$$const = -\frac{2}{\gamma-1} a_0$$

Rewrite the expression for u

$$u = \frac{2}{\gamma - 1} \frac{a_0}{\rho_0^{(\gamma-1)/2}} \rho^{(\gamma-1)/2} - \frac{2}{\gamma - 1} a_0$$

$$u = \frac{2}{\gamma - 1} \left(\boxed{a_0 \left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{2}}} - a_0 \right)$$

Recall that

$$a = \boxed{a_0 \left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{2}}}$$

Thus

$$u = \frac{2}{\gamma - 1} \left(a - a_0 \right)$$

The same expression with respect to a:

$$a = \frac{\gamma - 1}{2} u + a_0$$

The general solution was of the form...

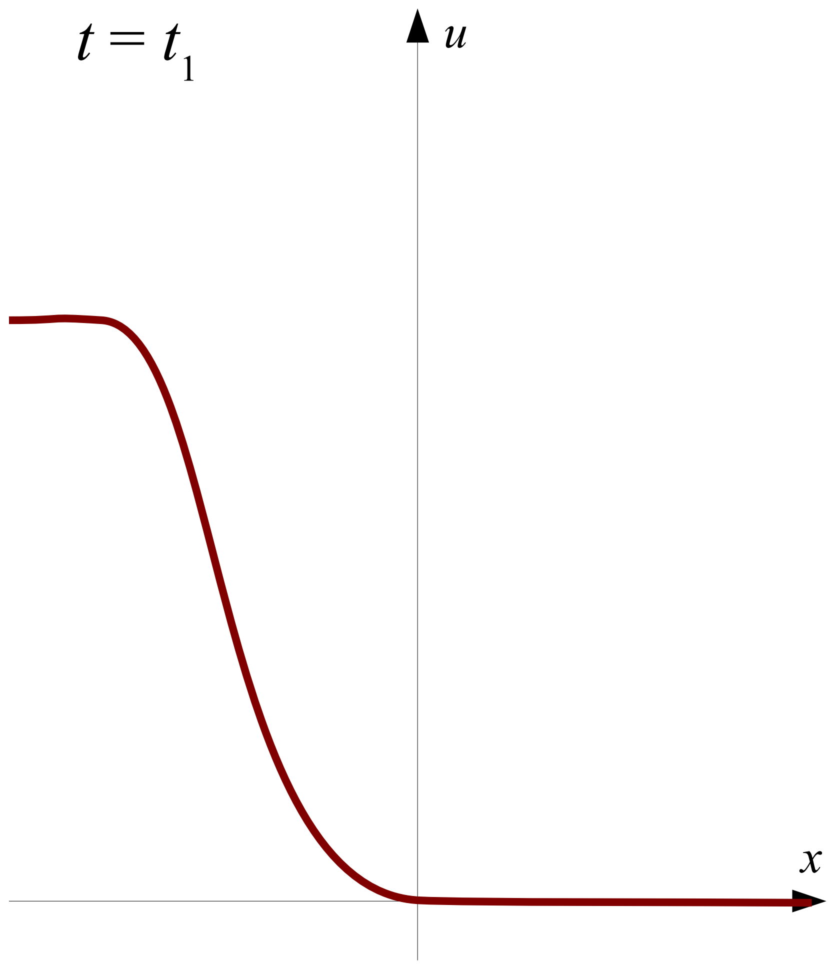
$$u = f\left(x - \underbrace{\left(u + a\right)}_{\text{Wave speed}} t\right)$$

For wave speed of the forward wave,

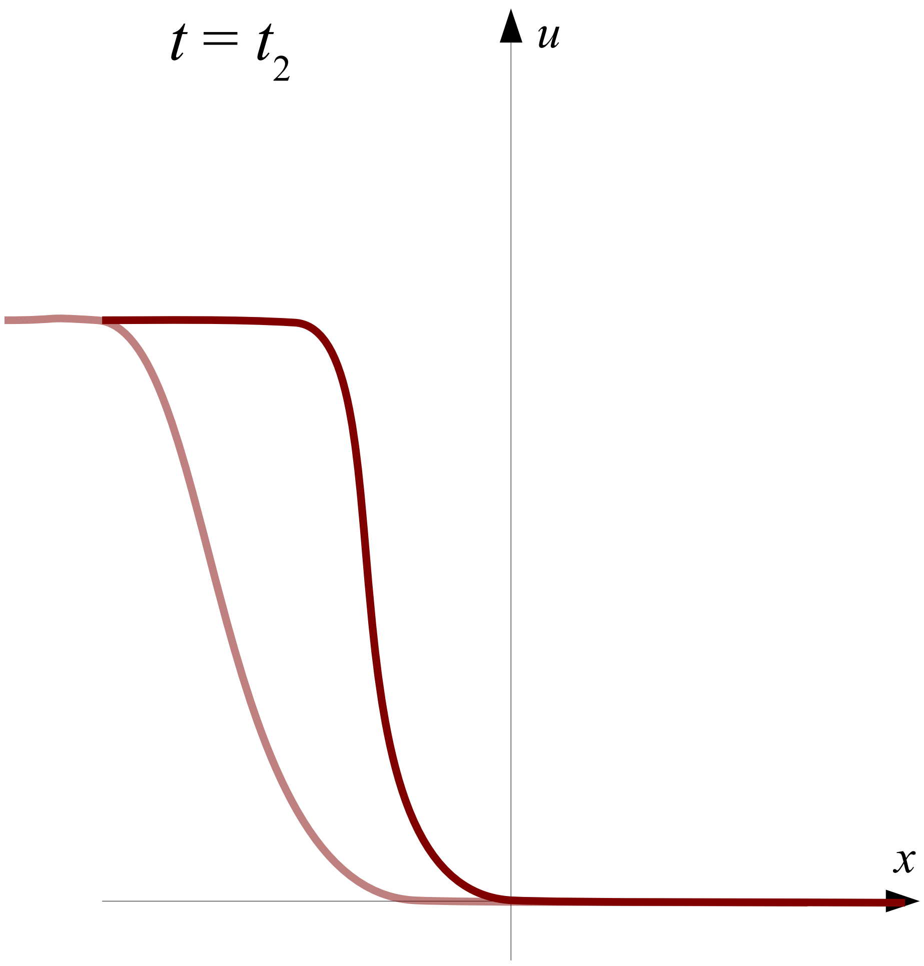
$$U = u + a = u\left(1 + \frac{\gamma - 1}{2}\right) + a_0 = a_0 + \frac{\gamma + 1}{2} u$$

The faster the local speed, the faster the wave wants to go!

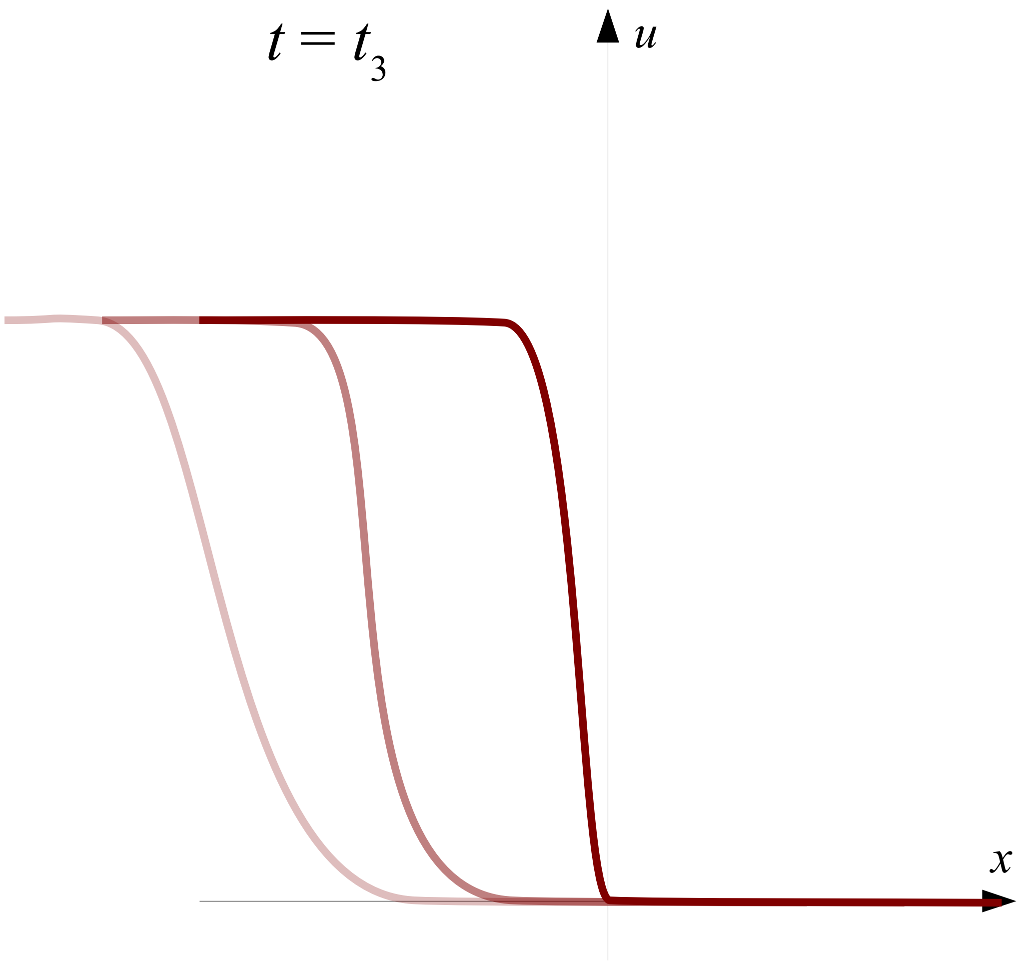
Evolution of a finite-amplitude velocity disturbance



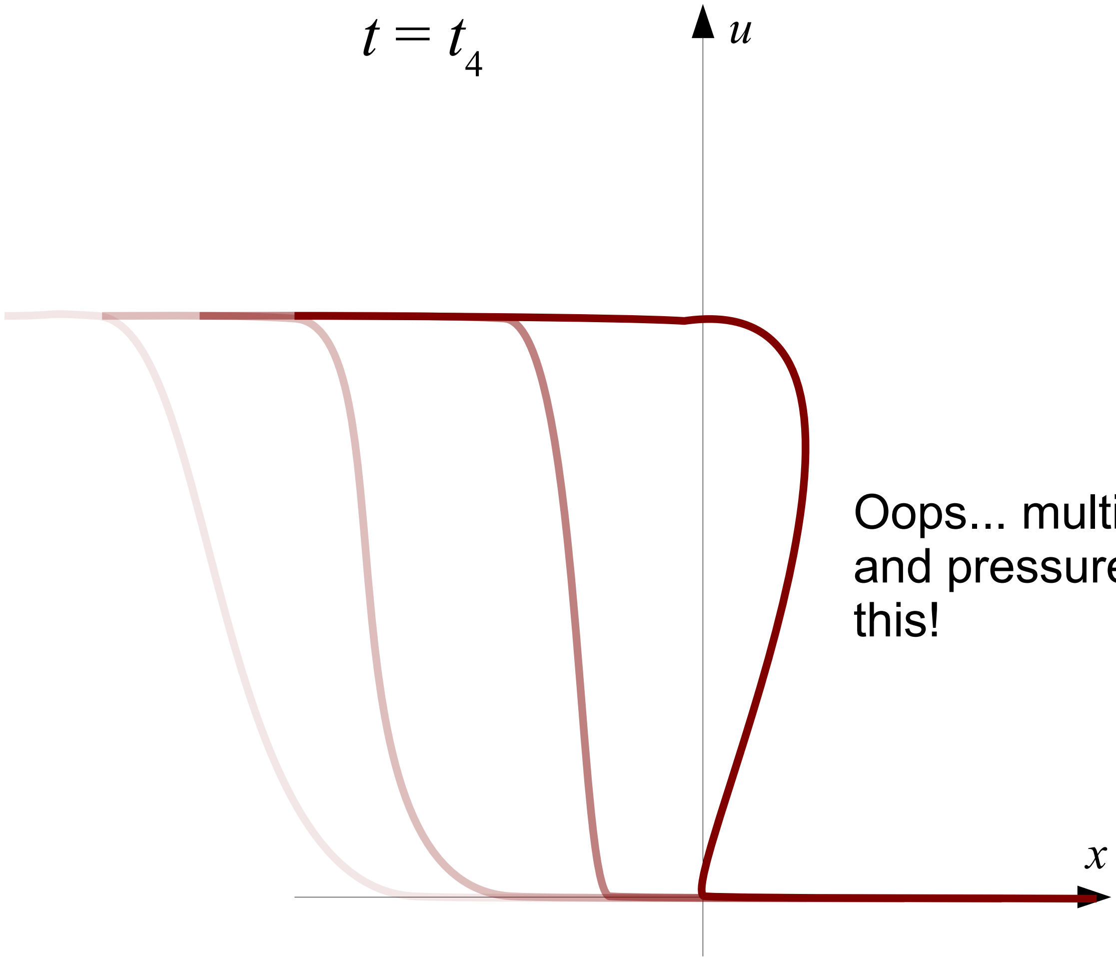
Evolution of a finite-amplitude velocity disturbance



Evolution of a finite-amplitude velocity disturbance



Evolution of a finite-amplitude velocity disturbance



Oops... multivalued velocity and pressure... cannot have this!

Note: this problem arises only for a positive-pressure perturbation (a negative-pressure finite perturbation, a.k.a. *rarefaction wave*, will disperse nicely)

$\nabla \cdot$

What really happens to finite-strength positive-pressure perturbation in gas...

$p = p_s,\ \rho = \rho_s,\ u = u_s$

Piston velocity (less than U)

$p = p_0,\ \rho = \rho_0,\ u = 0$

Ernst Mach
1838-1916

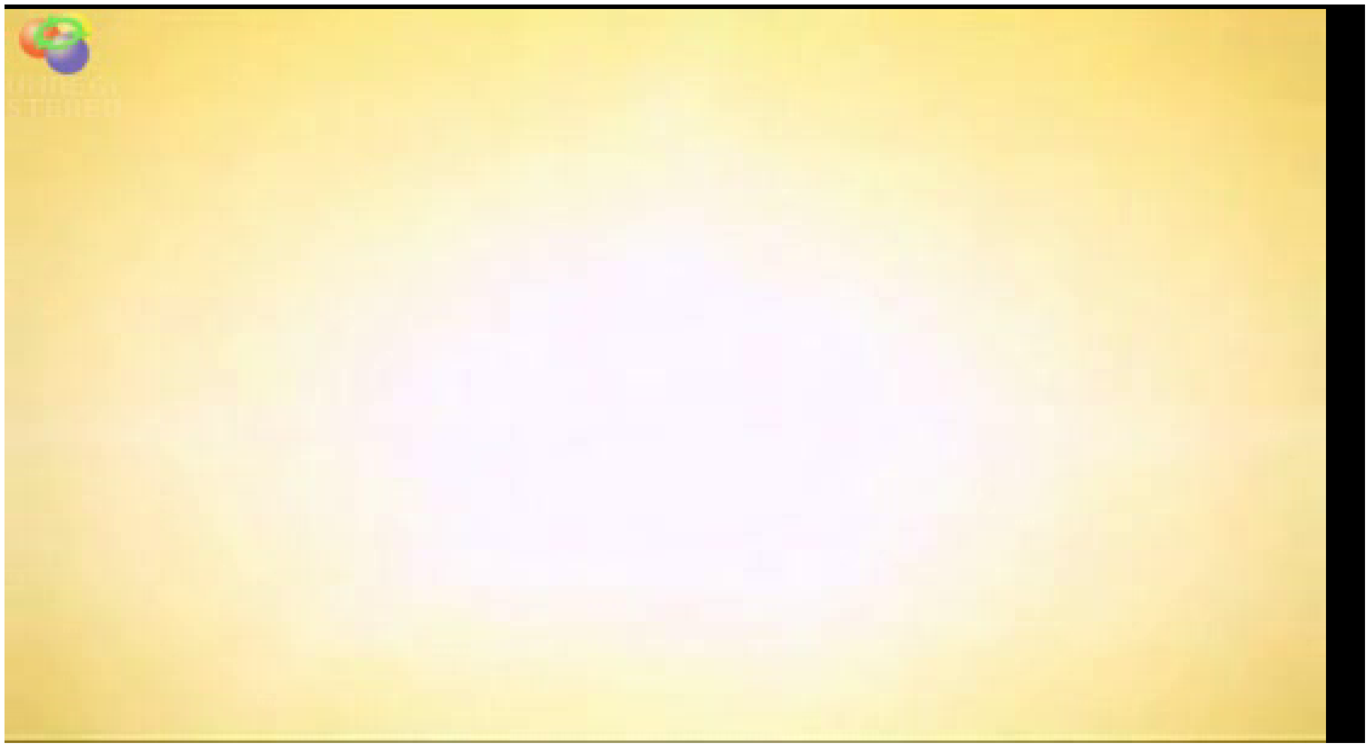
Shock front speed U

Mach number
$M = U/a_0 > 1$

Shock front

Here continuum approximation breaks down!

Explosion of Tsar-Bomba (AH602, ~60 megaton thermonuclear device), Kurchatov, Khariton, Sakharov



Explosion of Tsar-Bomba (AH602, ~60 megaton thermonuclear device), Kurchatov, Khariton, Sakharov

M65 atomic cannon test (Upshot-Knothole test series, 1953)

M65 atomic cannon test (Upshot-Knothole test series, 1953)

11.3. Rankine-Hugoniot equations

$p = p_1, \rho = \rho_1, u = u_1$ $p = p_2, \rho = \rho_2, u = u_2$

Normal shock
~~(as opposed to crazy mad shock)~~

(velocity normal to shock front)

Pierre Henri Hugoniot (1851-1887)

Reference frame moves with shock front (discontinuity in the framework of continuum theory, width ~ 1 mean free path of a molecule)

Shock front

Conservation equations

$$\rho_1 u_1 = \rho_2 u_2$$

Mass

Momentum jump on the interface due to pressure difference

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

Momentum

Momentum fluxes

Energy equation

$$\frac{u_1^2}{2} + c_{p1} T_1 = \frac{u_2^2}{2} + c_{p2} T_2$$

Rewrite enthalpy per unit mass $c_p T$ using ideal gas equation $p = \rho R T$:

$$c_p T = c_p \frac{p}{\rho R} = c_p \frac{p}{\rho(c_p - c_v)} = \frac{c_p / c_v}{c_p / c_v - 1} \frac{p}{\rho} = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}$$

With that, energy equation becomes

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2}$$

Divide momentum equation by mass equation...

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \qquad \rho_1 u_1 = \rho_2 u_2$$

$$u_1 + \frac{p_1}{\rho_1 u_1} = u_2 + \frac{p_2}{\rho_2 u_2}$$

Rearrange

$$u_2 - u_1 = \frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = \frac{p_1 - p_2}{\rho_1 u_1}$$

$\rho_1 u_1 = \rho_2 u_2$

(mass equation)

Multiply this by $(u_2 + u_1)$

$$u_2^2 - u_1^2 = \frac{p_1 - p_2}{\rho_1 u_1}(u_2 + u_1) = \frac{p_1 - p_2}{\rho_1}\left(\frac{u_2}{u_1} + 1\right)$$

Divide mass equation by $\rho_2 u_1$

$$\frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}$$

Use this to replace u_2/u_1 in energy equation

$$u_2^2 - u_1^2 = \frac{p_1 - p_2}{\rho_1}\left(\frac{\rho_1}{\rho_2} + 1\right) = \left(p_1 - p_2\right)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)$$

Rewrite an earlier form of energy equation to get

$$u_2^2 - u_1^2 = 2\left(\frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} - \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}\right)$$

With a little tidying up...

$$2\frac{\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right)=\left(p_1-p_2\right)\left(\frac{1}{\rho_2}+\frac{1}{\rho_1}\right)$$

Multiply by ρ_2

$$2\frac{\gamma}{\gamma-1}\left(p_1\frac{\rho_2}{\rho_1}-p_2\right)=\left(p_1-p_2\right)\left(1+\frac{\rho_2}{\rho_1}\right)$$

Collect terms with density ratio, solve for it

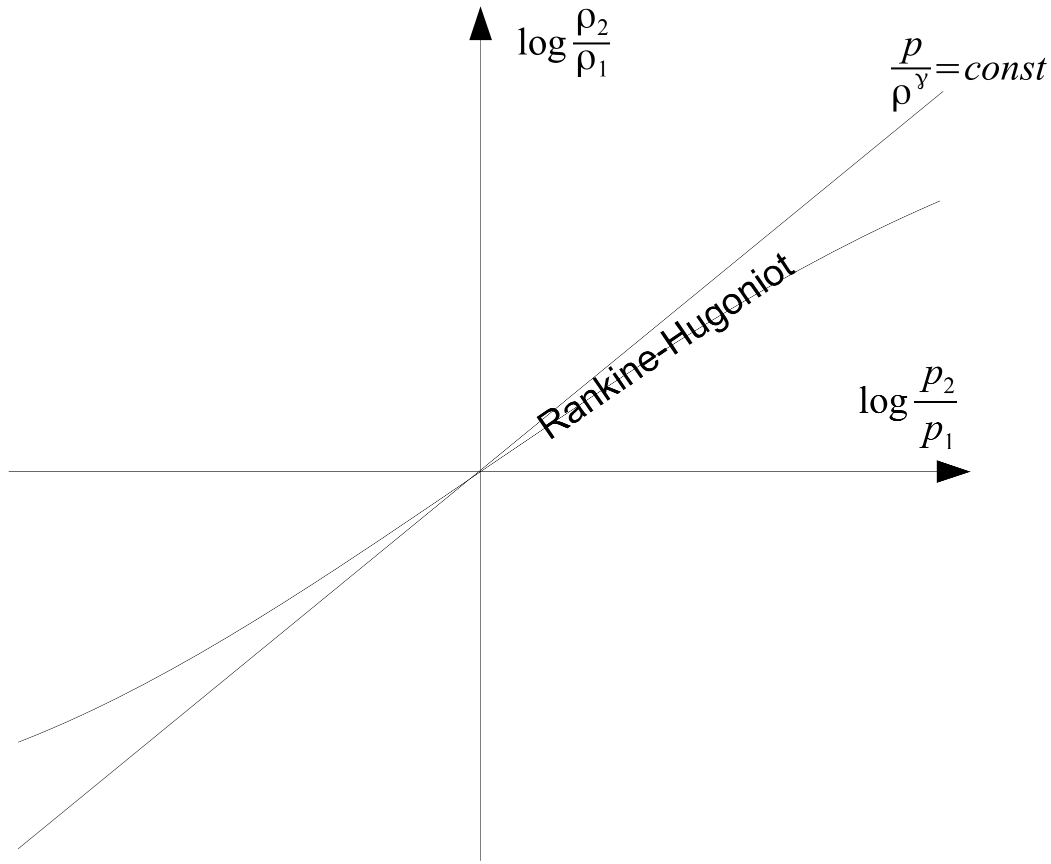
$$\frac{\rho_2}{\rho_1}=\frac{u_1}{u_2}=\frac{p_1+\frac{\gamma+1}{\gamma-1}p_2}{\frac{\gamma+1}{\gamma-1}p_1+p_2}$$

Rankine-Hugoniot equations

From energy eq.

11.4. Conditions for normal shock waves

Across the shock, the flow is **not isentropic**



This part cannot be realized - violates 2nd law of thermodynamics

With more algebra (using the expression we had for velocity difference...)

$$u_2 u_1 = a_*^2$$

Prandtl-Meyer relation

We will be able to use it, but first...
How exactly does the second law of thermodynamics apply at the shock front?

Calorically perfect gas (Appendix E.5)

Perfect gas equation of state

$$p = \rho R T$$

Specific heat at constant volume

$$C_v = \left[\frac{dq}{dT}\right]_v = \frac{\partial e}{\partial T} = \frac{\partial h}{\partial T} + \left[\frac{\partial h}{\partial p} - v\right]\left[\frac{\partial p}{\partial T}\right]_v$$

(Enthalpy $h = e + pv$)

Specific heat at constant pressure

$$C_p = \left[\frac{dq}{dT}\right]_p = \frac{\partial h}{\partial T} = \frac{\partial e}{\partial T} + \left[\frac{\partial e}{\partial v} + p\right]\left[\frac{\partial v}{\partial T}\right]_p$$

Perfect gas equation of state

$$p = \rho R T \quad \Leftrightarrow \quad pv = RT$$

$$pdv + vdp = R\,dT$$

$$dh = d\left(e + pv\right) = de + pdv + vdp$$

$$dh = de + RdT$$

$$\frac{\partial h}{\partial T} - \frac{\partial e}{\partial T} = R$$

From previous slide,

$$C_v = \frac{\partial e}{\partial T}, \quad C_p = \frac{\partial h}{\partial T}$$

Thus

$$R = C_p - C_v$$

Can further show that for perfect gas

$$e = e(T) = \int C_v \, dT + const$$

$$h = h(T) = \int C_p \, dT + const$$

Gas is called calorically perfect if

$$C_p = const\,, \quad C_v = const$$

Then for calorically perfect gas

$$e = e(T) = C_v T + const$$

$$h = h(T) = C_p T + const$$

Second law of thermodynamics (Appendix E.8)

Uniqiely determined by the state of the system

Entropy *s* – thermodynamic variable of state

Consider a system in equilibrium state *A*
By adding heat *Q* to the system, we move it to another equilibrium state *B*

$$Q = \int_A^B dq$$

Introduce entropy *s* so that the change in *s* between equilibrium states *A* and *B* is

$$s_B - s_A = \int_A^B \frac{dq}{T}$$

Evaluated for reversible process, i.e. change is so slow that system remains in thermodynamic equilibrium

Statement of the second law of thermodynamics

For any spontaneous process, the entropy change is non-negative

$$s_B - s_A \geqslant \boxed{\int_A^B \frac{dq}{T} \quad \text{Evaluated for reversible process}}$$

Consider calorically perfect gas

$$s_B - s_A = \left[C_p \log T - R \log p\right]_B - \left[C_p \log T - R \log p\right]_A$$

Now let states 1 and 2 correspond to ideal gas before and after the shock, then

$$s_2 - s_1 = \Delta s =$$

$$= \left[C_p \log T - R \log p \right]_2 - \left[C_p \log T - R \log p \right]_1 =$$

$$= C_p \log \frac{T_2}{T_1} - R \log \frac{p_2}{p_1}$$

For ideal gas, temperature $T = p/(\rho R)$, so

$$\Delta s = C_p \log \frac{p_2 \rho_1}{p_1 \rho_2} - R \log \frac{p_2}{p_1}$$

$$\Delta s = C_p \left(\log \frac{p_2}{p_1} + \log \frac{\rho_1}{\rho_2} \right) - R \log \frac{p_2}{p_1}$$

$$\Delta s = \boxed{\left(C_p - R\right)} \log \frac{p_2}{p_1} - C_p \log \frac{\rho_2}{\rho_1}$$

$$\Delta s = \boxed{C_v} \log \frac{p_2}{p_1} - C_p \log \frac{\rho_2}{\rho_1}$$

$$\frac{\Delta s}{C_v} = \log \frac{p_2}{p_1} - \gamma \log \frac{\rho_2}{\rho_1}$$

State 2 can be reached from state 1 by some kind of quasi-equilibrium, slow isentropic (I) process, then $s_2 = s_1$ and

$$\frac{\Delta s}{C_v} = 0 = \log \frac{p_2}{p_1} - \gamma \log \left[\frac{\rho_2}{\rho_1}\right]_I$$

We arrive at state 2 by shock acceleration (nearly instant, thus definitely not isentropic) – use subscript *RH*

$$\frac{\Delta s}{C_v} = \log \frac{p_2}{p_1} - \gamma \log \left[\frac{\rho_2}{\rho_1} \right]_{RH}$$

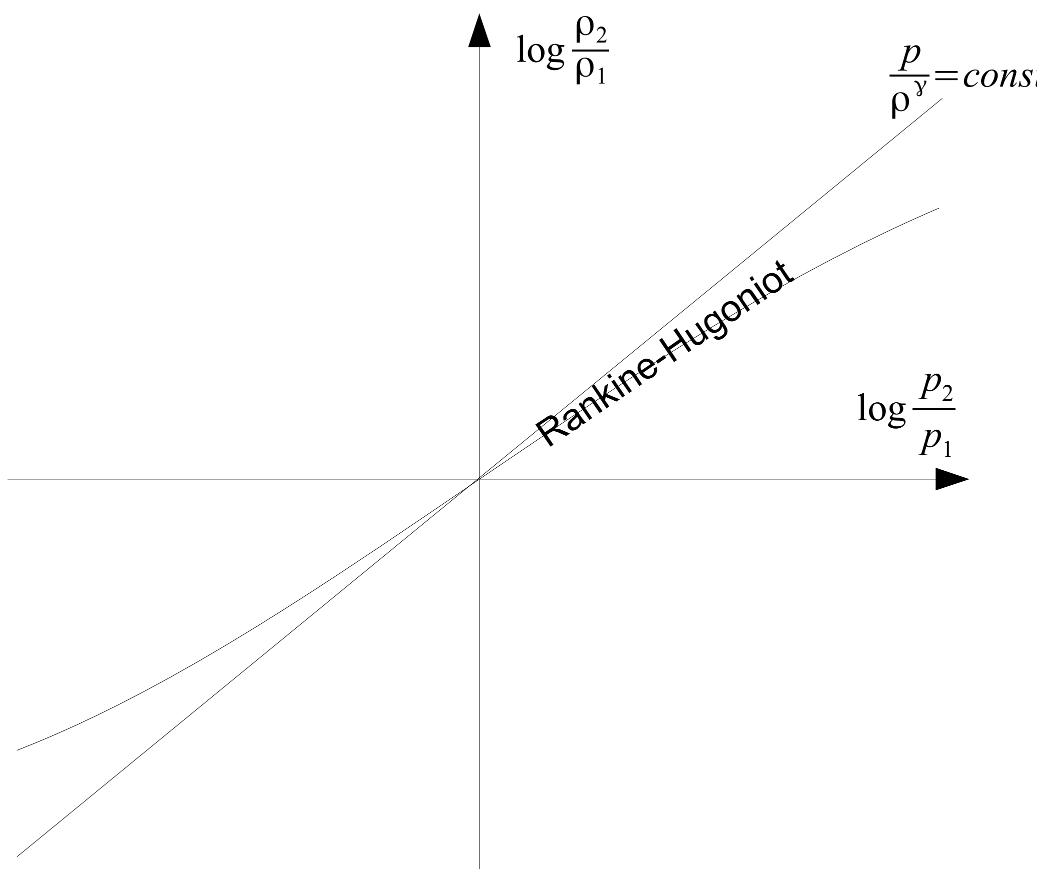
From previous slide, use

$$\log \frac{p_2}{p_1} = \gamma \log \left[\frac{\rho_2}{\rho_1} \right]_I$$

$$\frac{\Delta s}{C_v} = \gamma \log \left[\frac{\rho_2}{\rho_1} \right]_I - \gamma \log \left[\frac{\rho_2}{\rho_1} \right]_{RH}$$

From second law of thermodynamics,

$$\Delta s = C_v \gamma \left(\log\left[\frac{\rho_2}{\rho_1}\right]_I - \log\left[\frac{\rho_2}{\rho_1}\right]_{RH} \right) > 0$$



Here Δs would be negative

Some interesting corollaries

A shock wave is thermodynamically realizable if

$$\log\frac{\rho_2}{\rho_1} \geqslant 0, \quad \log\frac{p_2}{p_1} \geqslant 0$$

Recall that, from Rankine-Hugoniot equations,

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2}$$

Thus

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \geqslant 1$$

Definition for speed of sound from acoustics section

$$a_0 = \sqrt{\gamma R T_0} = \sqrt{\gamma \frac{p_0}{\rho_0}}$$

Let $a^2 = \gamma p/\rho$, use that to rewrite energy equation yet again

$$\frac{u_1^2}{2} + \frac{a_1^2}{\gamma - 1} = \frac{u_2^2}{2} + \frac{a_2^2}{\gamma - 1}$$

Use $*$ subscript to denote the case when $u = a$ ($M = 1$, sonic flow) and

$$\frac{u_*^2}{2} + \frac{a_*^2}{\gamma - 1} = a_*^2\left(\frac{1}{2} + \frac{1}{\gamma - 1}\right) = \frac{\gamma + 1}{2(\gamma - 1)} a_*^2$$

With more algebra (using the expression we had for velocity difference...)

$$u_2 u_1 = a_*^2$$

Prandtl-Meyer relation

From the second law of thermodynamics, we had

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \geqslant 1$$

Rewrite this as

$$\frac{u_1^2}{u_1 u_2} \geqslant 1 \quad \text{or} \quad \frac{u_1^2}{a_*^2} \geqslant 1$$

Earlier we had energy equation in the form...

$$\frac{u_1^2}{2}+\frac{a_1^2}{\gamma-1}=\frac{u_2^2}{2}+\frac{a_2^2}{\gamma-1}$$

We formally introduced $*$ case from the right-hand side of the same energy equation if $M = 1$,

$u = a = u_* = a_*$

$$\frac{u_*^2}{2}+\frac{a_*^2}{\gamma-1}=\frac{\gamma+1}{2\left(\gamma-1\right)}a_*^2$$

Take energy equation in the form

$$\frac{u_1^2}{2}+\frac{a_1^2}{\gamma-1}=\frac{\gamma+1}{2\left(\gamma-1\right)}a_*^2$$

Divide both parts by u_1^2

$$\frac{1}{2}+\underbrace{\frac{a_1^2}{u_1^2}}_{M_1^2}\frac{1}{\gamma-1}=\frac{\gamma+1}{2(\gamma-1)}\frac{a_*^2}{u_1^2}$$

$$\frac{a_*^2}{u_1^2}=\left(\frac{1}{2}+\frac{1}{M_1^2(\gamma-1)}\right)\frac{2(\gamma-1)}{\gamma+1}$$

$$\frac{a_*^2}{u_1^2}=\left(\frac{M_1^2(\gamma-1)+2}{2\,M_1^2(\gamma-1)}\right)\frac{2(\gamma-1)}{\gamma+1}$$

Flip it over...

$$\frac{u_1^2}{a_*^2} = \frac{M_1^2(\gamma+1)}{M_1^2(\gamma-1)+2} \geqslant 1$$

$$M_1^2(\gamma+1) \geqslant M_1^2(\gamma-1)+2$$

$$M_1^2 \geqslant -M_1^2+2$$

$$M_1 \geqslant 1$$

Moreover, from Prandtl-Meyer relation it follows that

$$M_2 \leqslant 1$$