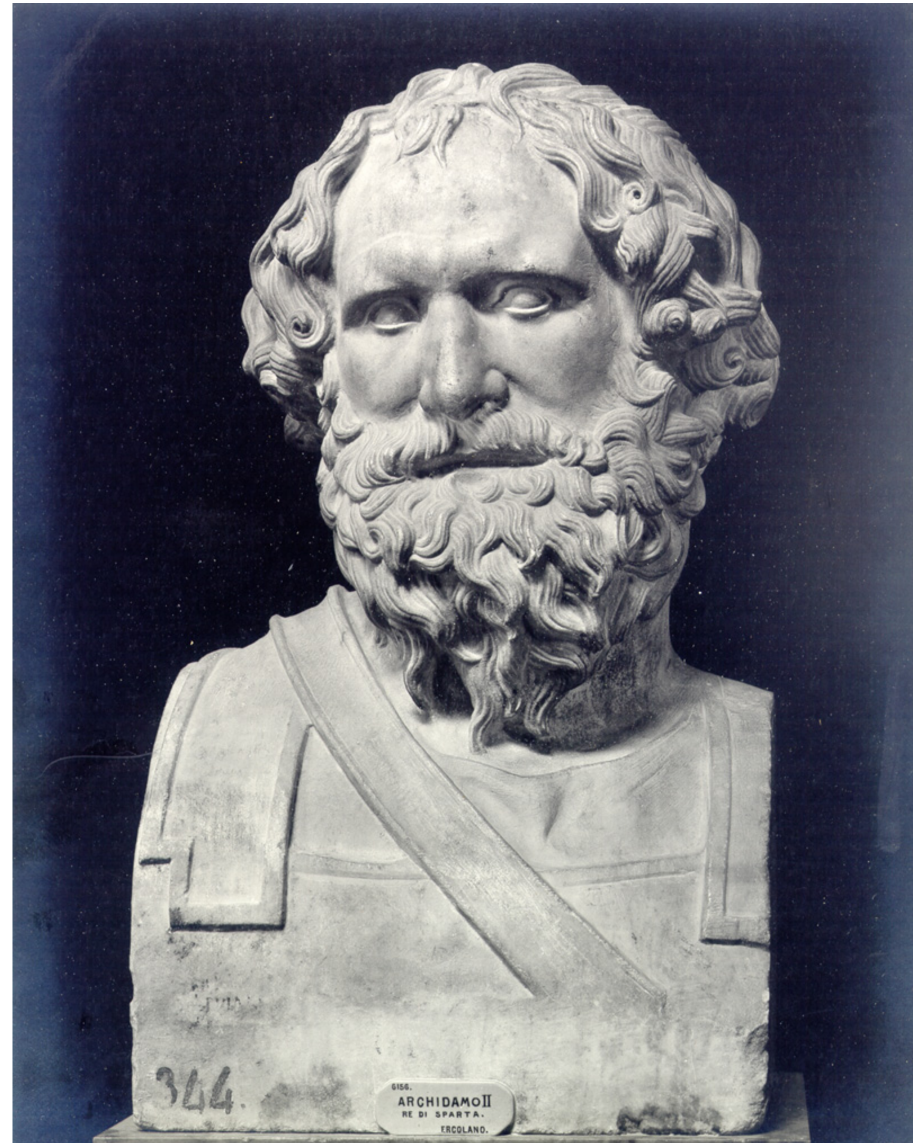
10. Buoyancy-driven flow

- For such flows to occur, need:
  - Gravity field
  - Variation of density (note: not the same as variable density!)
- Simplest case:
  - Viscous flow, incompressible fluid, density-variation effects only present in body force term



Archimedes, c. 287 BCE – c. 212 BCE

- Natural convection
  - Movement is due to buoyancy
  - Important forces:
    - Viscous
    - Buoyancy
  - Density variation mostly relevant for body-force term only
- Forced convection
  - Movement is due to other forces
  - Important forces:
    - Viscous
    - Buoyancy
    - Other
  - Density - unknown
- Need energy equation and equations of state to close the system!

10.1. Boussinesq approximation

- Incompressible Navier-Stokes with gravity as body force

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \left(\boldsymbol{u} \cdot \nabla \right) \boldsymbol{u} = -\nabla p + \mu \nabla^2 \boldsymbol{u} - \rho g \, \boldsymbol{e}_z$$

Joseph Valentin Boussinesq (1842-1929)

- Hydrostatics ($\boldsymbol{u} = 0$, $p = p_0$, $\rho = \rho_0$):

$$0 = -\nabla p_0 - \rho_0 g \, \boldsymbol{e}_z$$

- Buoyancy-driven convective motion:

$$\boldsymbol{u} = \boldsymbol{u}^*, \; p = p_0 + p^*, \; \rho = \rho_0 + \rho^*$$

Plug expressions for p, ρ, $\boldsymbol{u}$ into momentum equation:

$$\left(\rho_0+\cancel{\rho^*}\right)\frac{\partial \boldsymbol{u}^*}{\partial t}+\left(\rho_0+\cancel{\rho^*}\right)\left(\boldsymbol{u}^*\cdot\nabla\right)\boldsymbol{u}^*=$$

$$-\boldsymbol{\nabla}\left(p^*+\boxed{p_0}\right)+\mu\nabla^2\boldsymbol{u}^*-\left(\boxed{\rho_0}+\rho^*\right)g\,\boldsymbol{e}_z$$

Throw out hydrostatic terms

Linearize, assuming

$$\rho_0 \gg \rho^*, \quad \rho=\rho_0+\rho^*\approx\rho_0$$

Drop * and $_0$, replace $\rho^* = \Delta\rho$

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u}=-\boldsymbol{\nabla}\,p+\mu\nabla^2\boldsymbol{u}-\Delta\rho\,g\,\boldsymbol{e}_z$$

Boussinesq approximation to momentum equation

10.2. Thermal convection

Fluid density change due to small temperature variations:

$$\rho = \rho_0 \left(1 - \beta \left(T - T_0\right)\right)$$

Thermal expansion coefficient

(valid for $\rho \approx \rho_0$, $T \approx T_0$)

For ideal gas, $\beta = 1/T_0$, so Boussinesq equation becomes

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \left(\boldsymbol{u} \cdot \nabla\right) \boldsymbol{u} = -\boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{u} - \rho g \beta \left(T - T_0\right) \boldsymbol{e}_z$$

ρ is not variable, but still need energy equation!

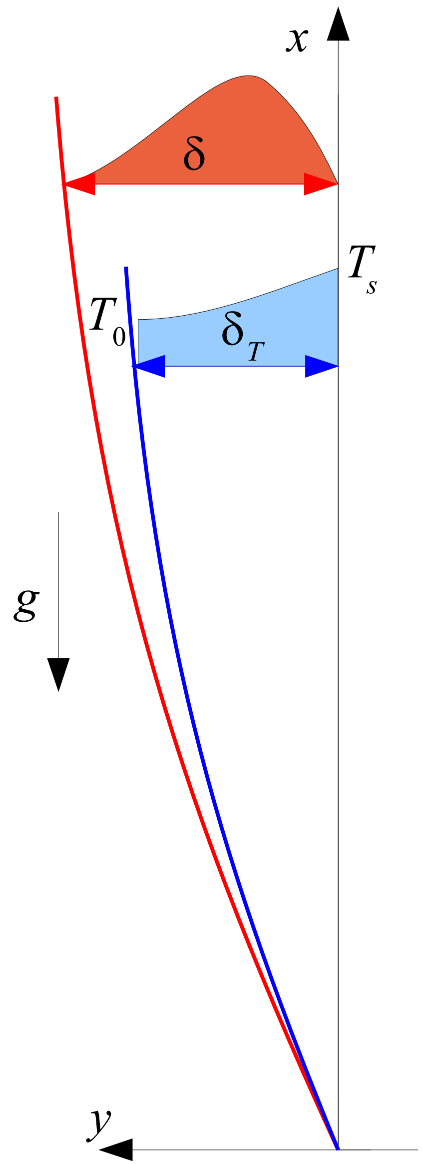
Note: Boussinesq approximation assumes incompressible fluid of constant density in momentum and continuity equations, but body force is due to density variation!
(self-contradictory assumption, but works quite well for finite variations of density with temperature!)

10.3. Boundary-layer approximation

Away from surface:

$T = T_0$

$U = 0$



On the surface:

$T = T_s$ (or other condition)

$U = 0$

BL thicknesses: thermal δ_T and velocity δ

- Three possible cases (both for natural and forced convection):
  - $\delta_T << \delta$: velocity in thermal BL small, heat transfer dominated by conduction, energy equation uncouples
  - $\delta_T >> \delta$: temperature gradient in velocity BL small, can assume it plays negligible role, energy equation uncouples
  - $\delta_T \sim \delta$: have to solve energy equation with momentum and continuity to get profiles for natural convection (Prandtl number ~ 1!)

Consider $\delta_T \sim \delta$

Full energy equation (1.14)

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = -p \frac{\partial u_k}{\partial x_k} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j} \right) + \Phi$$

$$\Phi = \lambda \left(\frac{\partial u_k}{\partial x_k} \right)^2 + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_j}{\partial x_i}$$

Dissipation function

Rewrite in terms of enthalpy $h = e + p/\rho$ ($\rho = const$)

$$\rho \frac{\partial}{\partial t}\left(h - \frac{p}{\rho}\right) + \rho u_k \frac{\partial}{\partial x_k}\left(h - \frac{p}{\rho}\right) =$$

$$-p \frac{\partial u_k}{\partial x_k} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) + \Phi$$

Open brackets...

$$\rho \frac{\partial h}{\partial t} - \frac{\partial p}{\partial t} + \rho u_k \frac{\partial h}{\partial x_k} - u_k \frac{\partial p}{\partial x_k} =$$

$$-p \frac{\partial u_k}{\partial x_k} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) + \Phi$$

For incompressible fluid, this term is zero! (continuity)

$$\rho \frac{\partial h}{\partial t} + \rho u_k \frac{\partial h}{\partial x_k} = \frac{\partial p}{\partial t} + u_k \frac{\partial p}{\partial x_k} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j} \right) + \Phi$$

Steady flow, 2D, $k = const$

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \Phi$$

Let $h = c_p T$

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = u \frac{\partial p}{\partial x} + \cancel{v \frac{\partial p}{\partial y}} + k \left(\cancel{\frac{\partial^2 T}{\partial x^2}} + \frac{\partial^2 T}{\partial y^2} \right) + \Phi$$

finite

small

Terms of same order

Toss this term – thin BL, no y-variation of pressure

This << this

Equations we need to solve...

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{u}{\rho c_p}\frac{\partial p}{\partial x}+\kappa\frac{\partial^2 T}{\partial y^2}+\cancel{\Phi}$$

Thermal diffusivity $\kappa=\dfrac{k}{\rho c_p}$

For Φ, assume it's negligibly small (works for low viscosity, high Re - and all boundary layer flows are high-Re flows!)

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_0)$$

BL equations from Ch. 9

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Body-force term (Boussinesq approx.)

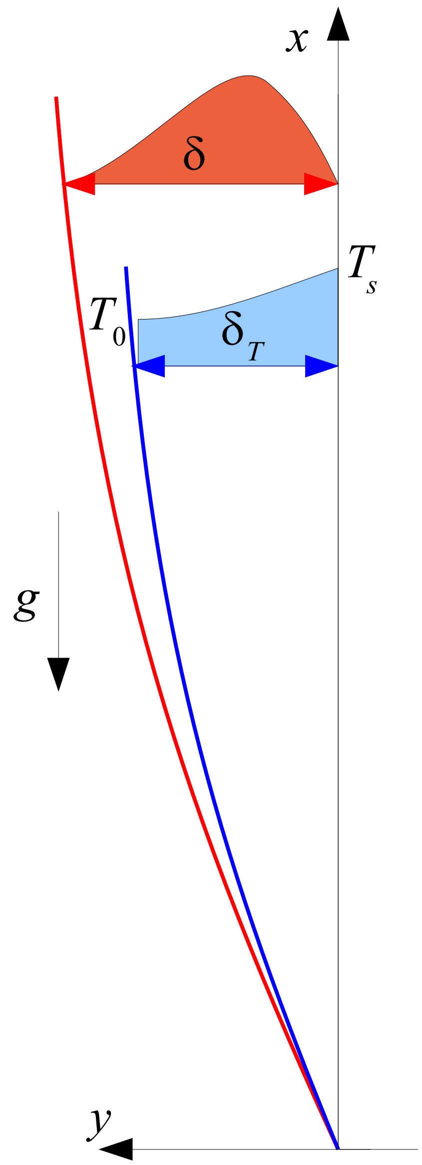
10.4. Vertical isothermal surface

Away from surface:

$T = T_0$

$U = 0$

$p = const$



On the surface:

$T = T_s$

$u = 0$

Thermal BL equations (zero pressure gradient)

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\kappa\frac{\partial^2 T}{\partial y^2}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta\left(T-T_0\right)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\big|_{y=0,\,y\to\infty}=0$$

Note – no condition on v at edge of BL (that would overdefine the system!)

$$v\big|_{y=0}=0$$

$$T\big|_{y=0}=T_s$$

$$T\big|_{y\to\infty}=T_0$$

Solution approaches

No length scale in x-direction – similarity solution

Can be simplified with polynomials (2nd order for T, 3rd for u) - Pohlhausen

Dimensionless parameters

Nusselt number

Convection coefficient

Length scale

$$Nu = \frac{hl}{k}$$

Grashof number

$$Gr = \frac{g\, l^3 \left(T_s - T_0\right)}{\nu^2 T_0}$$

Prandtl number

$$Pr = \frac{\nu}{\kappa}$$

$$Ra = Pr \cdot Gr$$

Rayleigh number

Physical meaning of dimensionless characteristics of thermal convection

Nu – how much more efficient convection (hl) is when compared to conduction (k); sometimes referred to as “dimensionless convection coefficient”

Gr – non-dimensional temperature differential driving the convection

Pr – measure of respective importance of mechanical (ν) to thermal (κ) dissipation

Ra – dimensionless buoyancy force

Pohlhausen's result for thermal convection in air ($Pr \sim 0.7$)

$$\frac{\delta}{x} = 5\left(\frac{Gr_x}{4}\right)^{-1/4}, \quad \delta \sim x^{1/4}$$

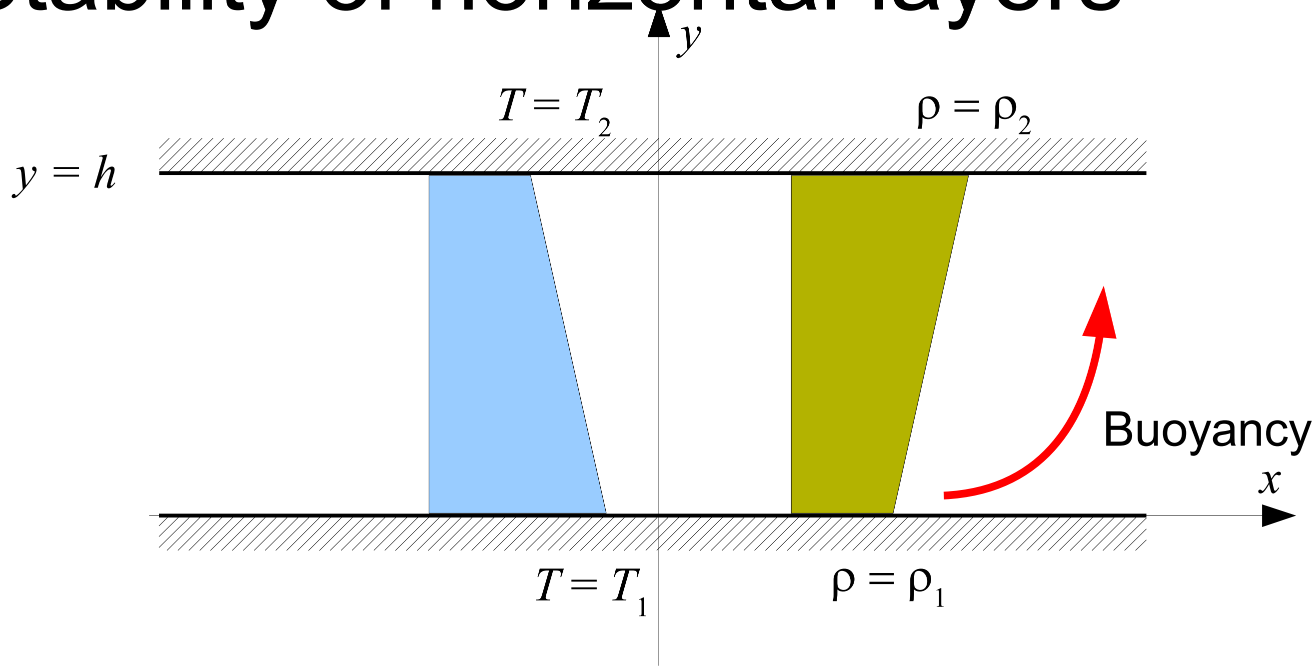
$$Nu \approx 0.359\, Gr^{1/4}$$

Thermal BL stability

$$Ra_{x,c} = \frac{g\beta\left(T_s - T_0\right)x^3}{\nu\kappa} \sim 10^9$$

Above $Ra_{x,c}$ – transition to turbulence!

10.7. Stability of horizontal layers



$T_1 < T_2$ (hot above cold) – stable (and stably stratified) horizontal thermal layer, heat transfer – by conduction only

$T_1 > T_2$ (cold above hot) – for some ΔT, unstable stratification leads to convection?

Governing parameter

$$Ra = \frac{g\beta\left(T_1 - T_2\right)h^3}{\nu\kappa}$$

Low Ra – no movement, heat transfer by conduction only (viscous forces >> buoyancy)

Consider small perturbation (velocity and temperature, 3D, time-dependent), look for Ra when any such perturbation can start growing

Thermal convection should start at

$$Ra > Ra_c = 1708$$

Great agreement with experiment!