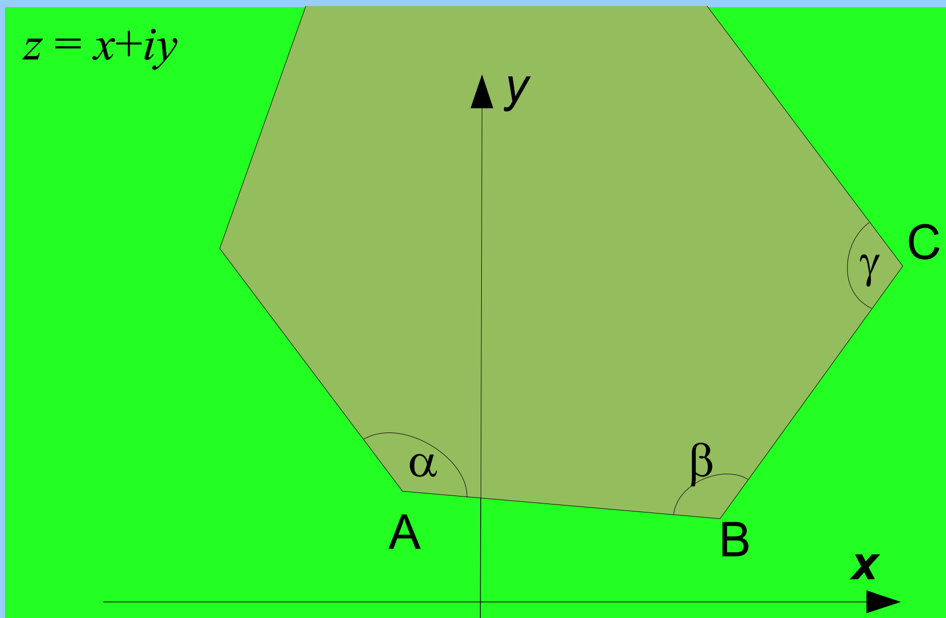
4.19. Schwarz-Christoffel transformation

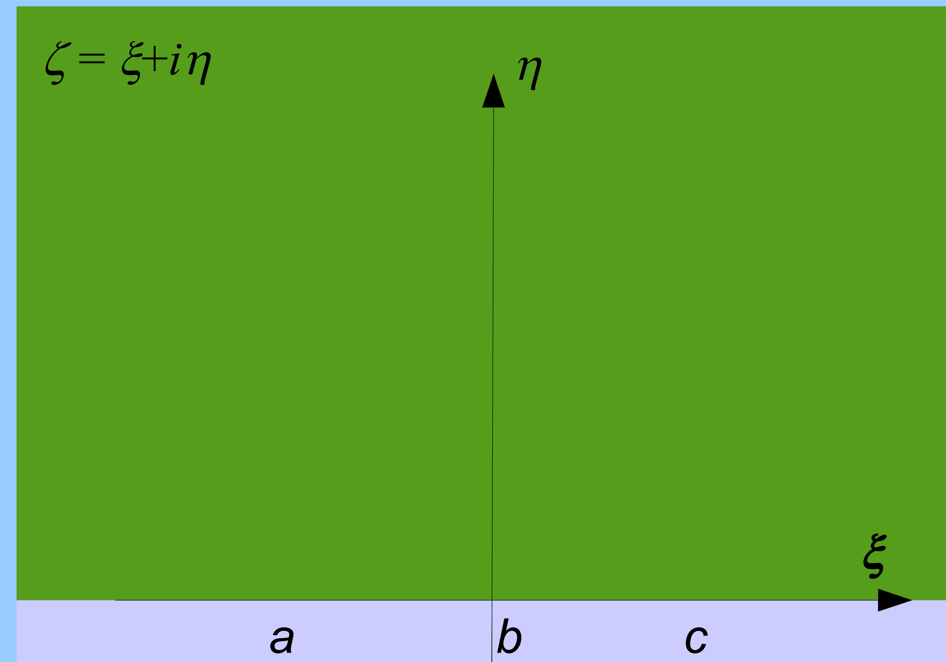
Elwin Bruno Christoffel
1829-1900

Hermann Amandus Schwarz
1843-1921

4.19. Schwarz-Christoffel transformation



Closed polygon:
Vertices A, B, C, … *n* total
Interior angles α, β, γ, … *n* total
Sum of all angles: $\pi(n\text{-}2)$



A $\rightarrow$ a
B $\rightarrow$ b
C $\rightarrow$ c

Schwarz-Christoffel transformation: differential form

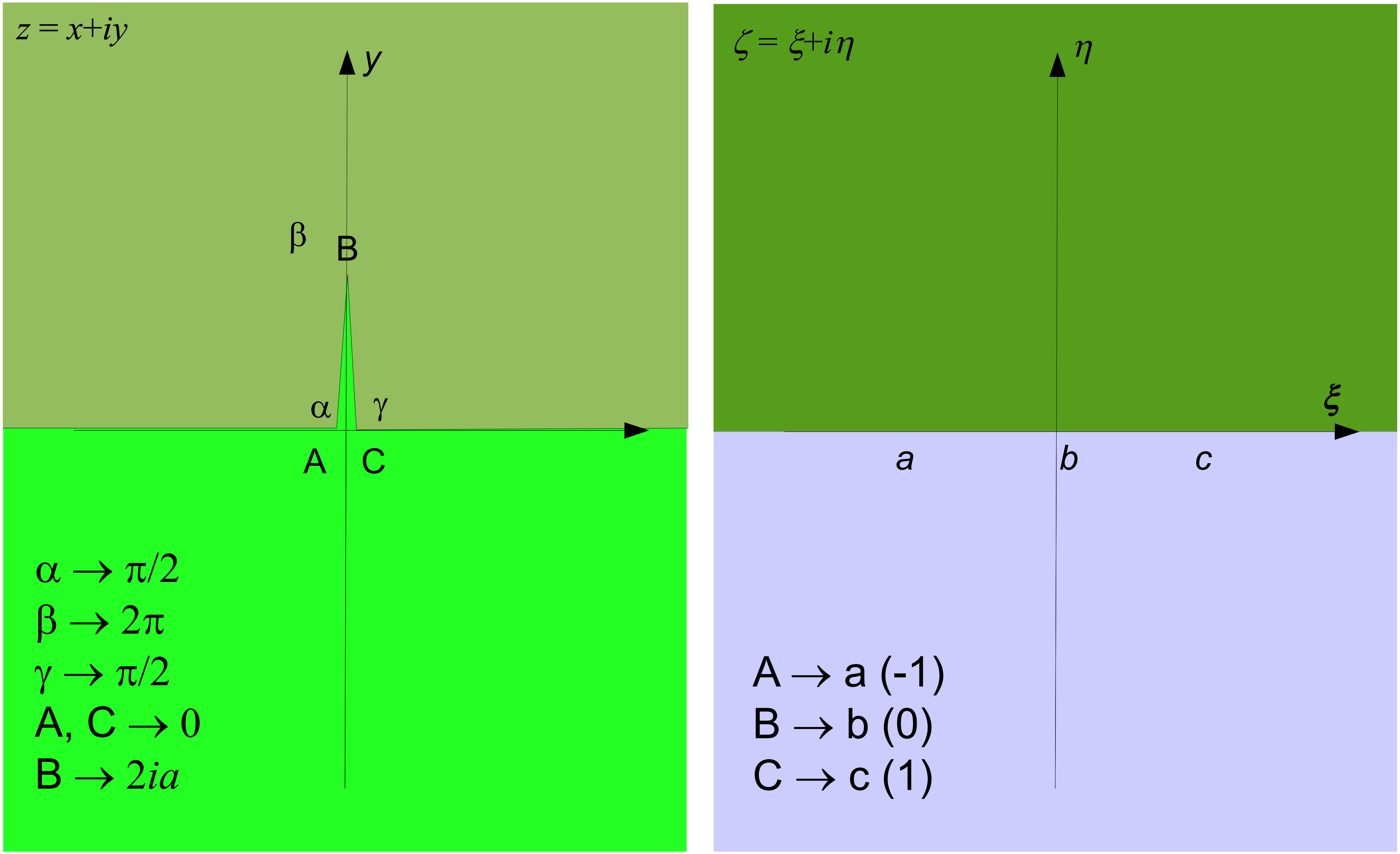
$$\frac{dz}{d\zeta} = K(\zeta - a)^{\frac{\alpha}{\pi}-1}(\zeta - b)^{\frac{\beta}{\pi}-1}(\zeta - c)^{\frac{\gamma}{\pi}-1}\ldots$$

Integral form

$$f(\zeta) = \int^{\zeta} \frac{K}{(\zeta' - a)^{1-\frac{\alpha}{\pi}}(\zeta' - b)^{1-\frac{\beta}{\pi}}(\zeta' - c)^{1-\frac{\gamma}{\pi}}\ldots} d\zeta'$$

For convenience, usually $a = -1$, $b = 0$, $c = 1$

Example 1. Flow past a vertical flat plate



$$\frac{dz}{d\zeta} = K(\zeta - a)^{\frac{\alpha}{\pi} - 1}(\zeta - b)^{\frac{\beta}{\pi} - 1}(\zeta - c)^{\frac{\gamma}{\pi} - 1}\ldots$$

$$\frac{dz}{d\zeta} = K(\zeta + 1)^{\frac{\pi/2}{\pi} - 1}\zeta^{\frac{2\pi}{\pi} - 1}(\zeta - 1)^{\frac{\pi/2}{\pi} - 1}$$

$$\frac{dz}{d\zeta} = K\frac{\zeta}{\sqrt{\zeta^2 - 1}}$$

Integrate...

$$z = K\sqrt{\zeta^2 - 1} + D$$

Let $K = 2a$, $D = 0$

Then

A$(-0) \rightarrow$ a (-1), B $(2ia) \rightarrow$ b (0), C $(+0) \rightarrow$ c (1)

$$z = 2a\sqrt{\zeta^2 - 1}$$

Expressing ζ through z,

$$\zeta = \pm\sqrt{\left(\frac{z}{2a}\right)^2 + 1}$$

Let $z \to$ +∞ as $\zeta \to$ +∞ (select + sign)

As $z, \zeta \to \infty$, $z \approx 2a\zeta$

We had...

$$w(z) = \frac{d\zeta}{dz} w(\zeta)$$

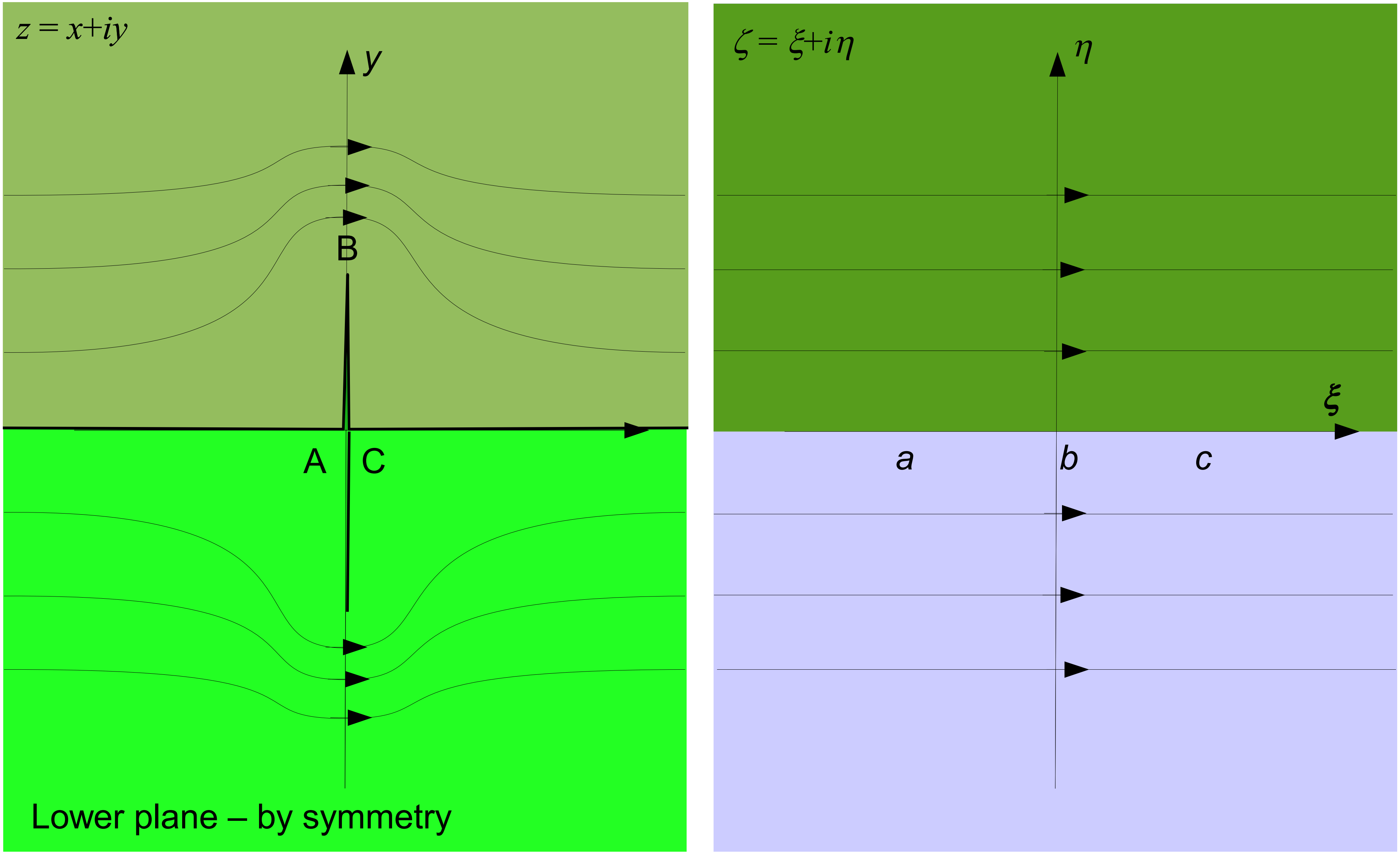
That's how velocity scales during conformal mapping

$$\lim_{z,\zeta \to \infty} \frac{d\zeta}{dz} = \frac{1}{2a}$$

Let complex potential $F(\zeta) = 2aU\zeta$

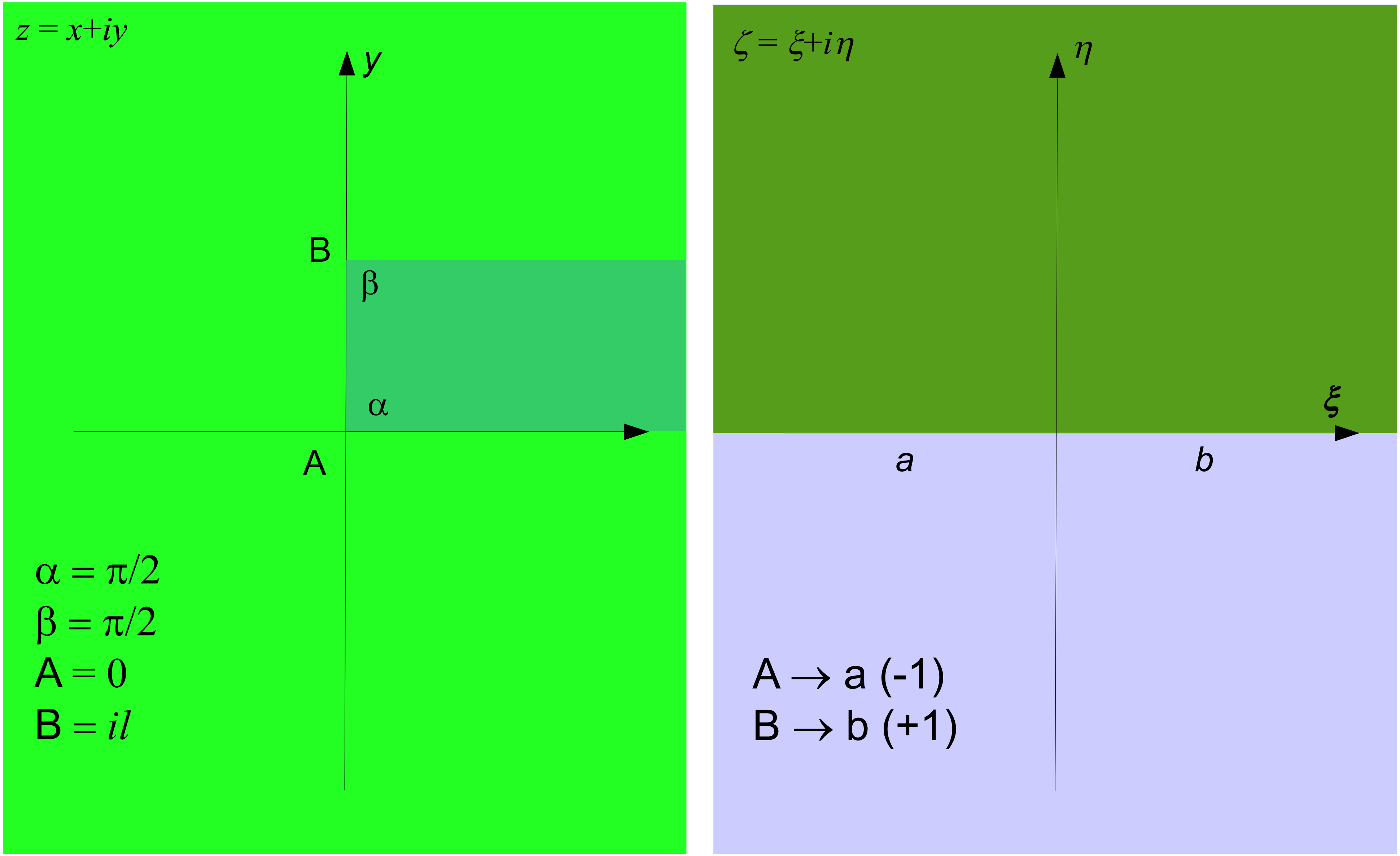
Then for $z, \zeta \to \infty$, $F(z) \to Uz$

$$F(\zeta) = 2aU\zeta = 2aU\sqrt{\left(\frac{z}{2\mathrm{a}}\right)^2 + 1} = U\sqrt{z^2 + 4a^2} = F(z)$$



z = x+iy
y
B
A
C
Lower plane – by symmetry
ζ = ξ+iη
η
ξ
a
b
c

4.20 Source in a channel (example 2)



$$\frac{dz}{d\zeta}=K(\zeta-a)^{\frac{\alpha}{\pi}-1}(\zeta-b)^{\frac{\beta}{\pi}-1}(\zeta-c)^{\frac{\gamma}{\pi}-1}\cdots$$

$$\frac{dz}{d\zeta}=K(\zeta+1)^{\frac{\pi/2}{\pi}-1}(\zeta-1)^{\frac{\pi/2}{\pi}-1}$$

$$\frac{dz}{d\zeta}=\frac{K}{\sqrt{\zeta^2-1}}$$

Integrate...

$$z=K\cosh^{-1}\zeta+D$$

Let $D = 0$, $K = l/\pi$,then

A(0) $\leftrightarrow$ a(-1)

B(il) $\leftrightarrow$ b(1)

$$z = \frac{l}{\pi} \cosh^{-1} \zeta$$

Again (as in previous example), express ζ through z

$$\zeta = \cosh \frac{\pi z}{l}$$

Now put a source in ζ-plane at point b ($\zeta = 1$)

$$F(\zeta) = \frac{m}{2\pi} \log(\zeta - 1)$$

Back in z-plane,

$$F(z) = \frac{m}{2\pi} \log\left(\cosh \frac{\pi z}{l} - 1 \right)$$

Trigonometric identity

$$\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta$$

Hyperbolic function identity

$$\cosh(\alpha+\beta)-\cosh(\alpha-\beta)=2\sinh\alpha\sinh\beta$$

Let $\alpha = \beta = \pi z/(2l)$

$$\cosh\frac{\pi z}{l}-1=\cosh\left(\frac{\pi z}{2l}+\frac{\pi z}{2l}\right)-\cosh\left(\frac{\pi z}{2l}-\frac{\pi z}{2l}\right)$$

$$\cosh\frac{\pi z}{l}-1=2\sinh^{2}\frac{\pi z}{2l}$$

Earlier we had

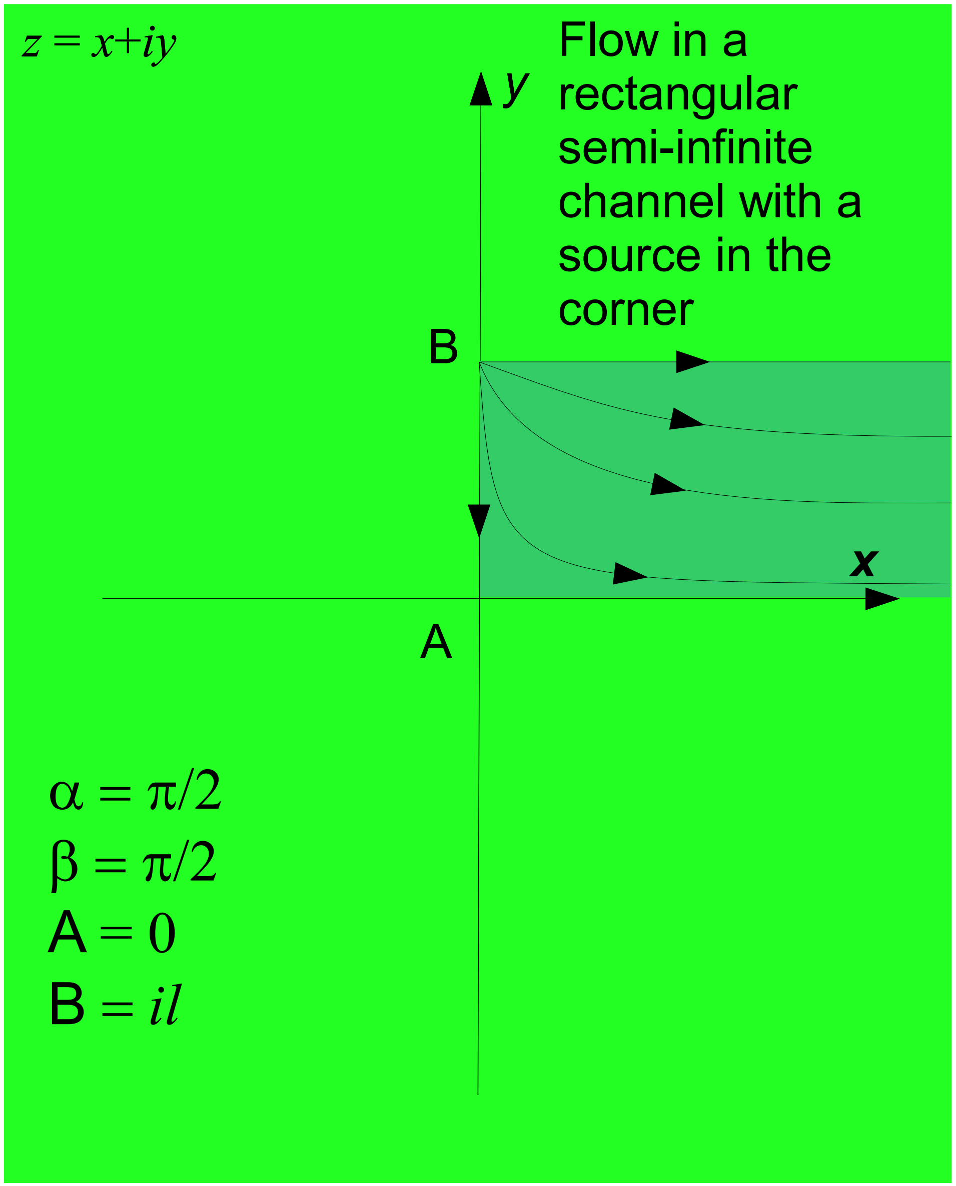
$$F(z) = \frac{m}{2\pi} \log\left(\cosh \frac{\pi z}{l} - 1\right)$$

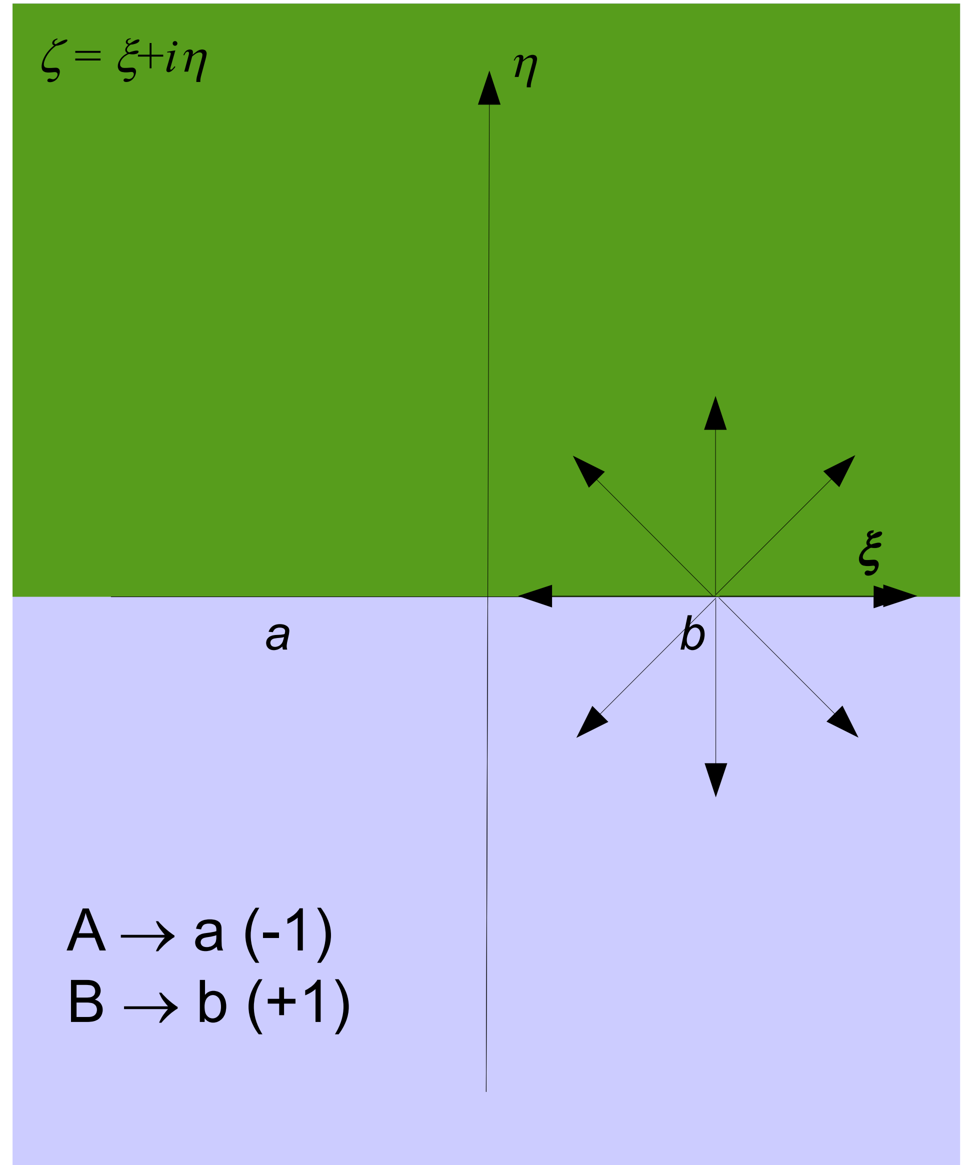
With the transformations,

$$F(z) = \frac{m}{2\pi} \log\left(2 \sinh^2 \frac{\pi z}{2l}\right)$$

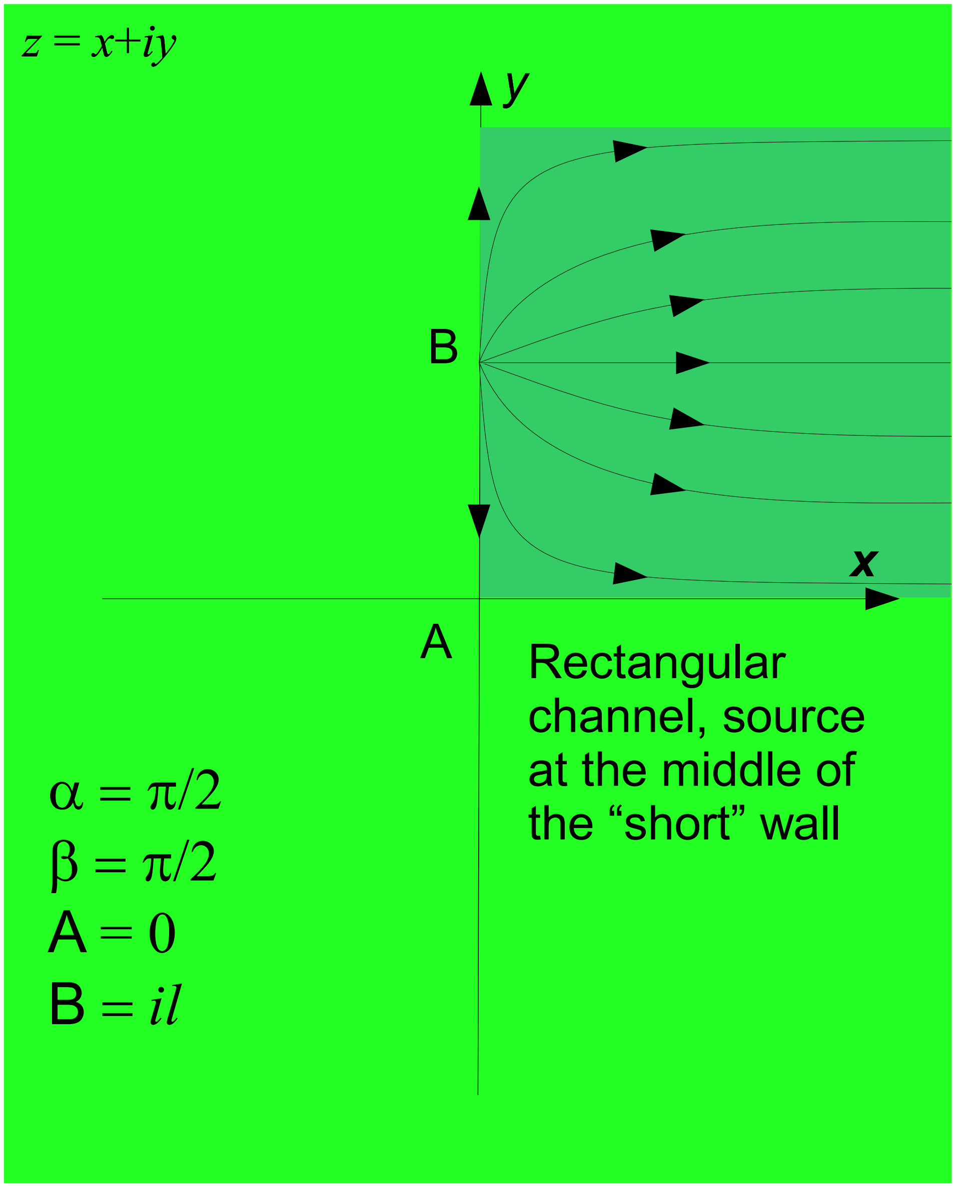
$$F(z) = \frac{m}{\pi} \log\left(\sinh \frac{\pi z}{2l}\right) + \cancel{\frac{m}{\pi} \log 2}$$

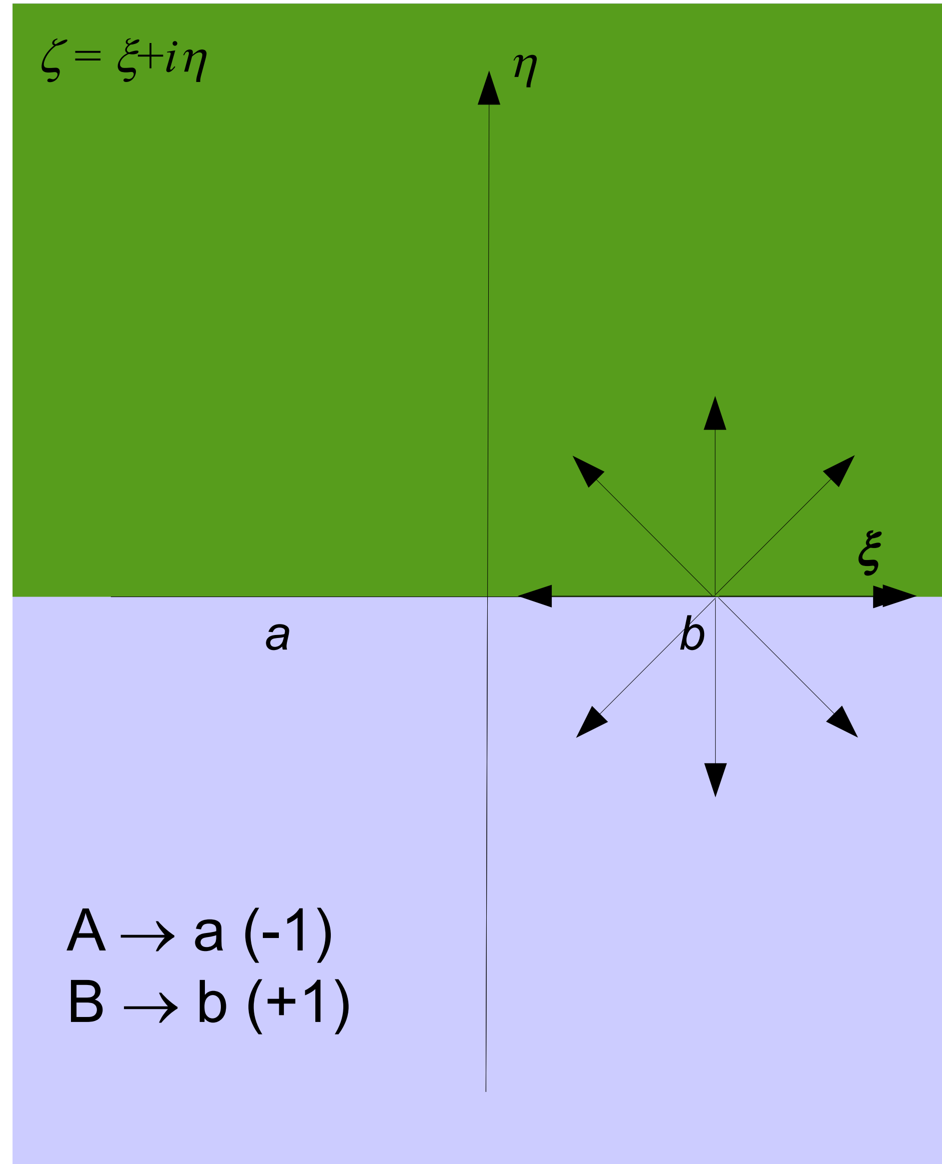
We don't care:
does not affect
velocity!


$z = x+iy$
Flow in a rectangular semi-infinite channel with a source in the corner
y
B
x
A
$\alpha = \pi/2$
$\beta = \pi/2$
A $= 0$
B $= il$

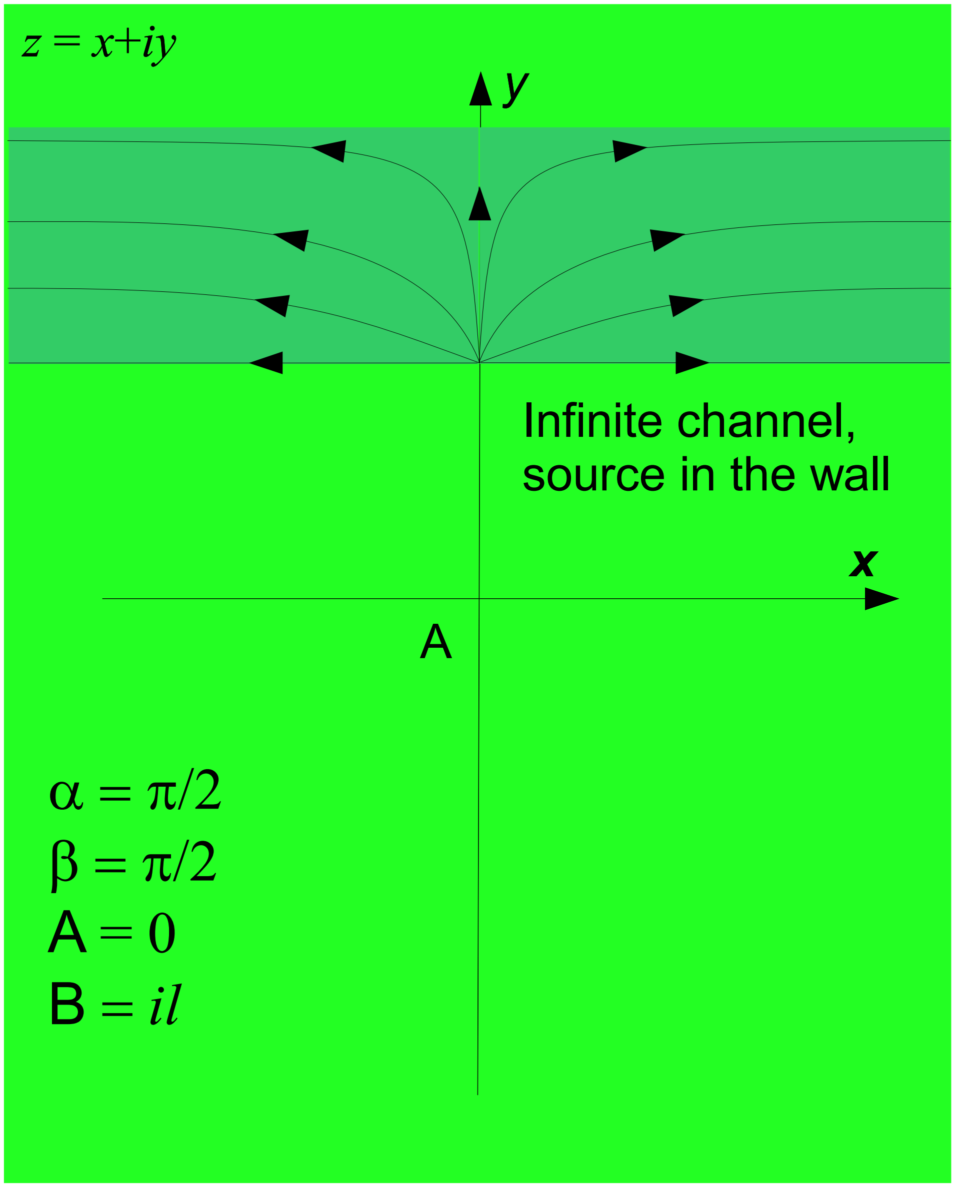

$\zeta = \xi+i\eta$
η
ξ
a
b
A $\rightarrow$ a (-1)
B $\rightarrow$ b (+1)

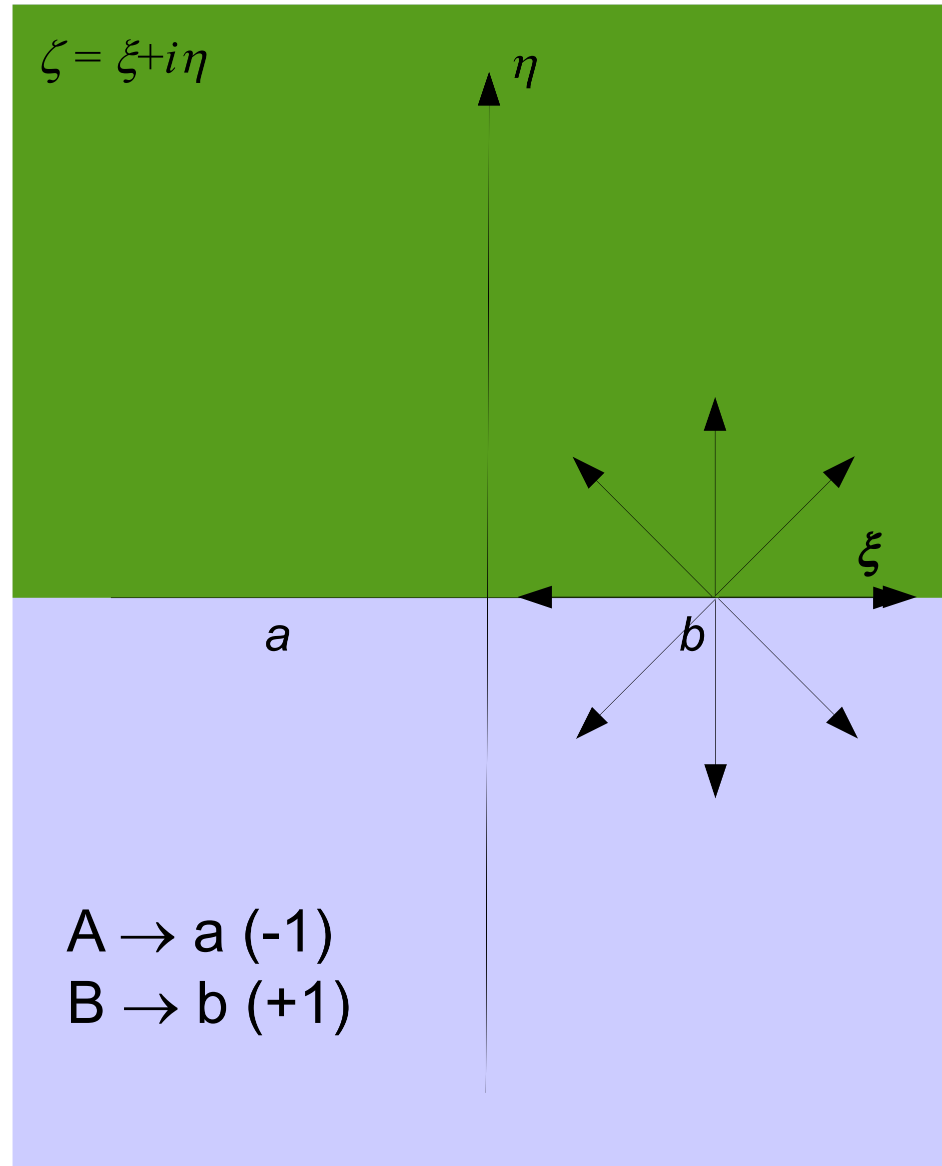
Use symmetry to create a variety of flows



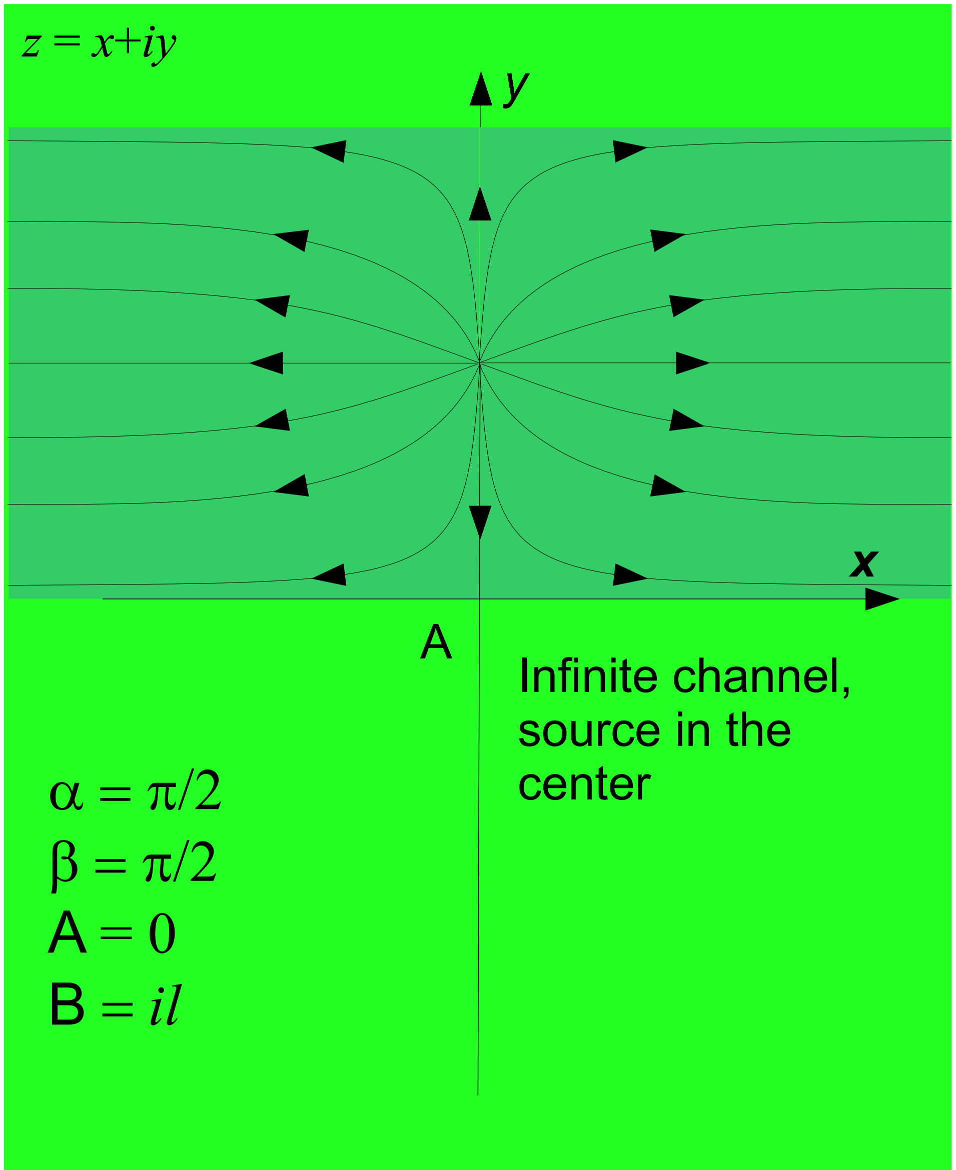


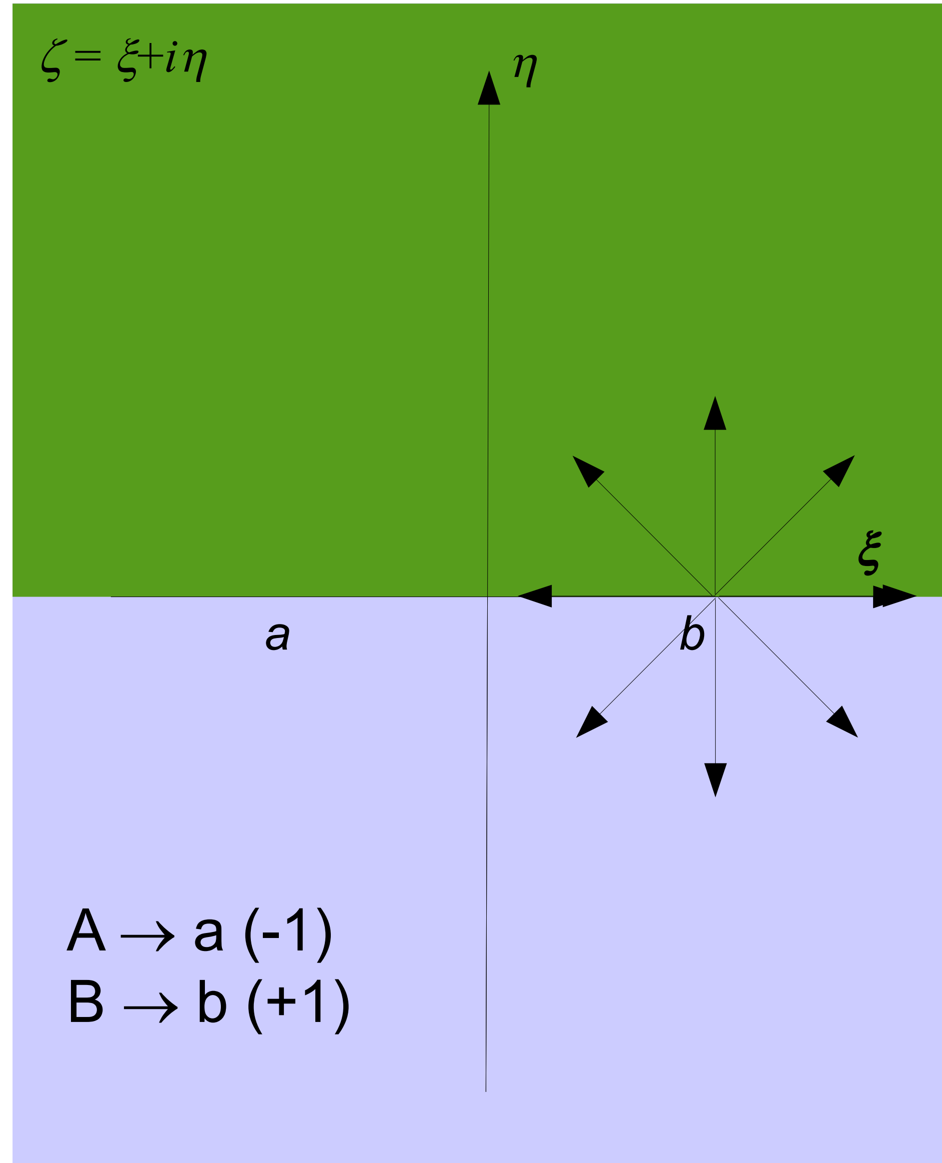
Use symmetry to create a variety of flows



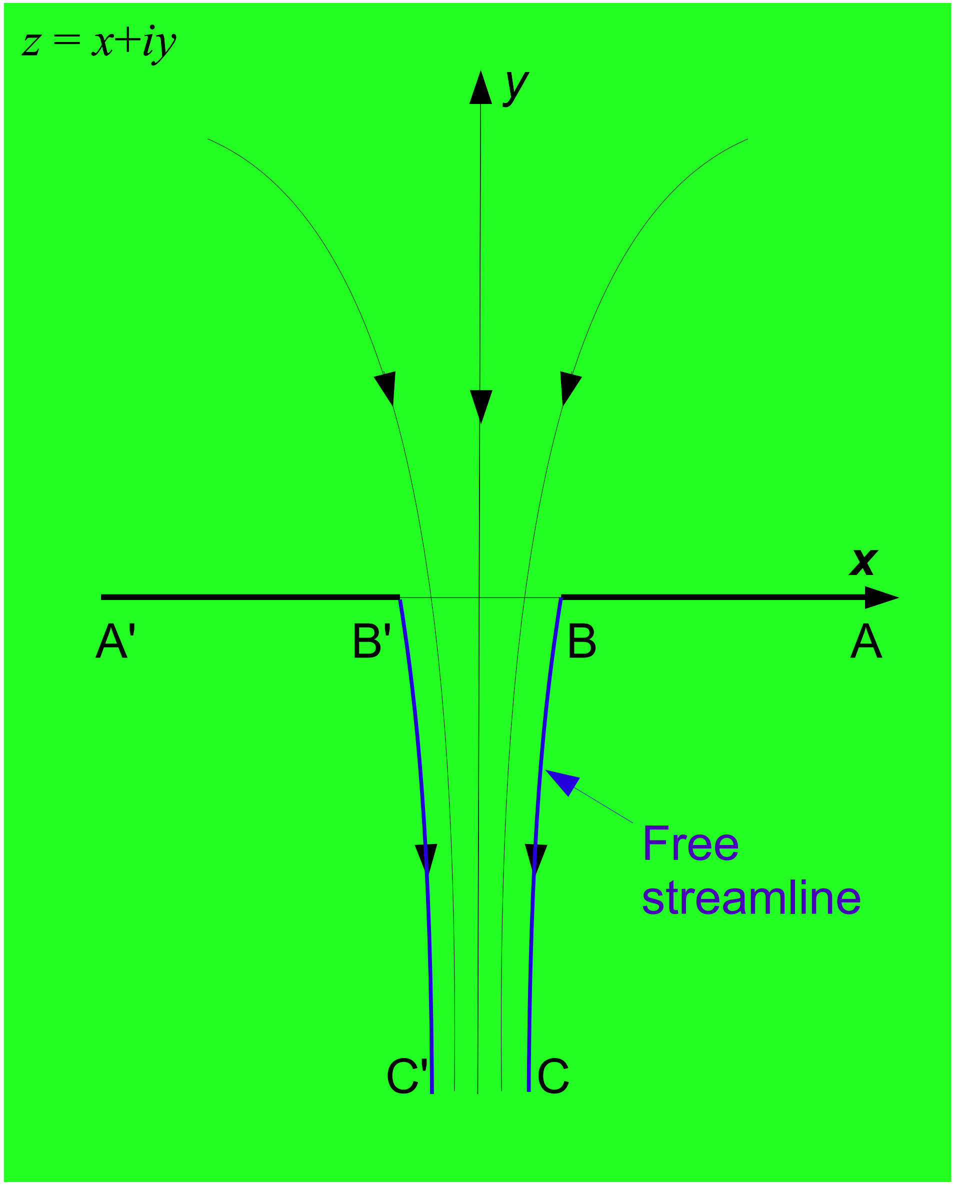


Use symmetry to create a variety of flows





4.21. Flow through an aperture



- Free streamlines:
  - Serve as boundaries of the fluid flow
  - Originate where flow separates from solid boundary
  - Exact shape may be unknown
  - Important condition on free streamline (not always, but often):

$$p = const$$

Free streamlines and hodograph plane

Bernoulli equation along a streamline (steady form, no mass forces)

$$\int \frac{dp}{\rho} + \frac{\boldsymbol{u}^2}{2} = const$$

Along a free streamline, $p = p_0$ (atmospheric pressure), thus

$$dp = 0 \text{ and } \boldsymbol{u}^2 = const$$

Now consider transformation to *hodograph* plane:

Characteristic velocity

$$\zeta = U \frac{dz}{dF} = \frac{U}{w} = \frac{U}{|w| e^{-i\theta}} = \frac{U}{\sqrt{u^2 + v^2}} e^{i\theta}$$

Along a free streamline,

$$u^2 + v^2 = const$$

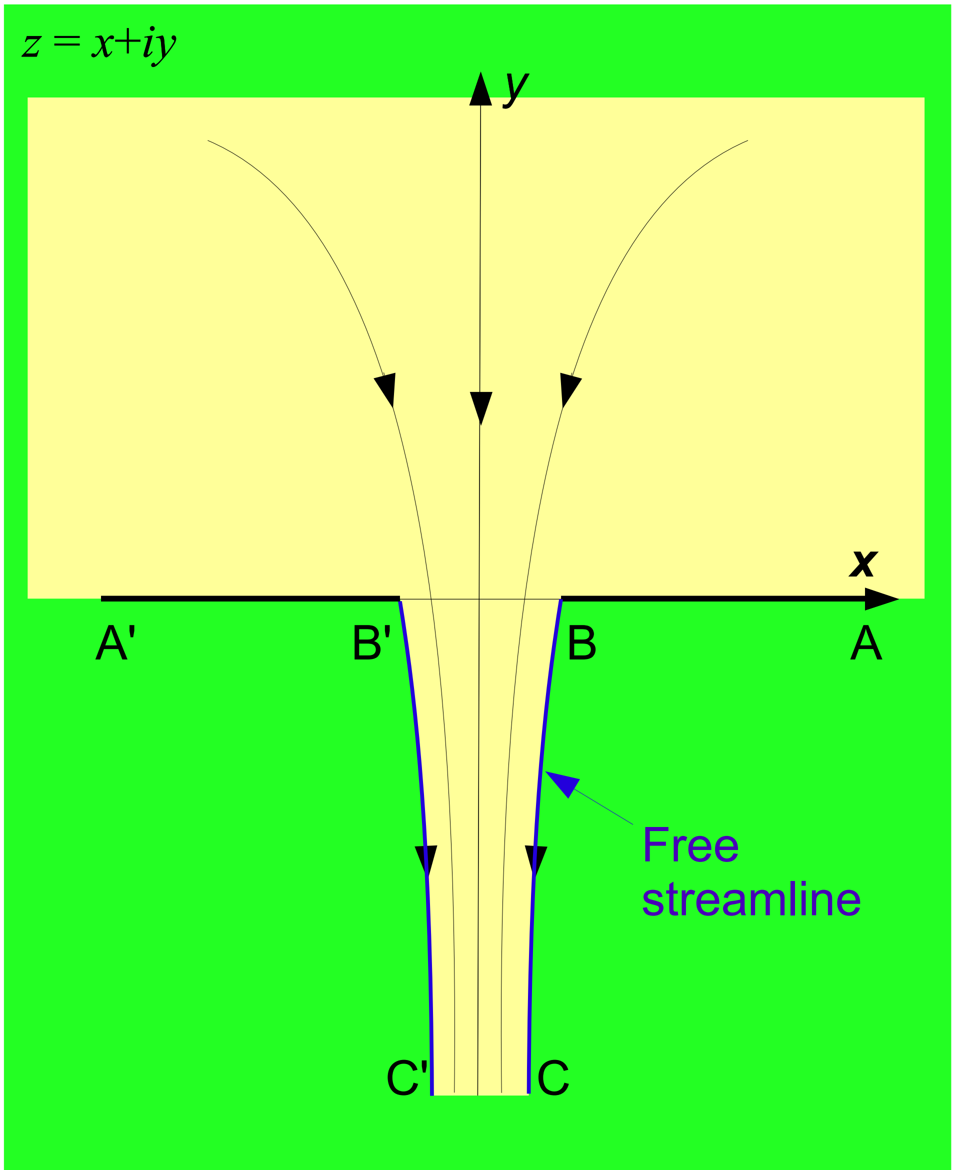
Select characteristic velocity so that

$$u^2 + v^2 = U^2$$

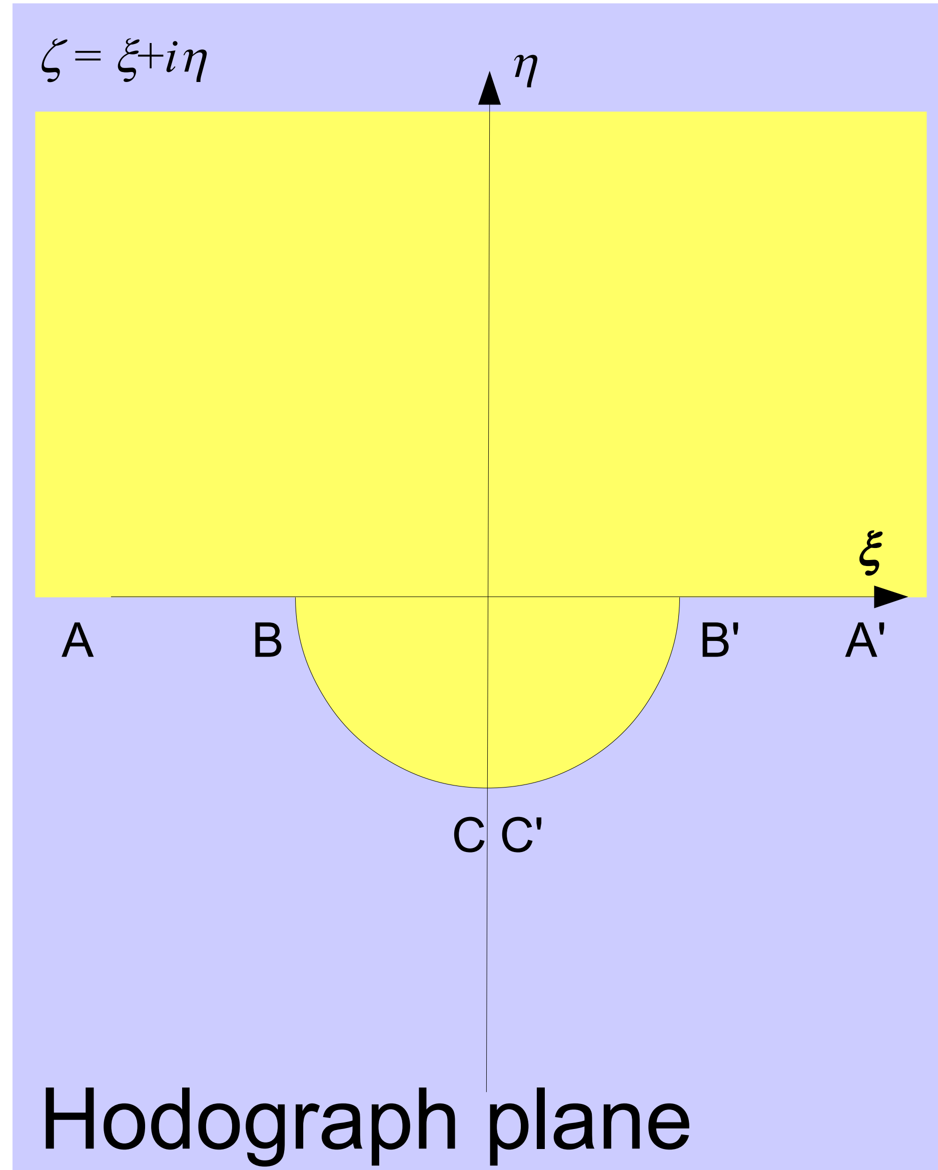
In the hodograph plane, a free streamline has a known shape...

$$\zeta = \frac{U}{\sqrt{u^2+v^2}} e^{i\theta} = e^{i\theta}$$

Unit circle



$z = x + iy$
y
x
A'
B'
B
A
C'
C
Free streamline



$\zeta = \xi + i\eta$
η
ξ
A
B
B'
A'
C C'
Hodograph plane

Remapping the hodograph plane

Need to map the flow area into something rectangular...

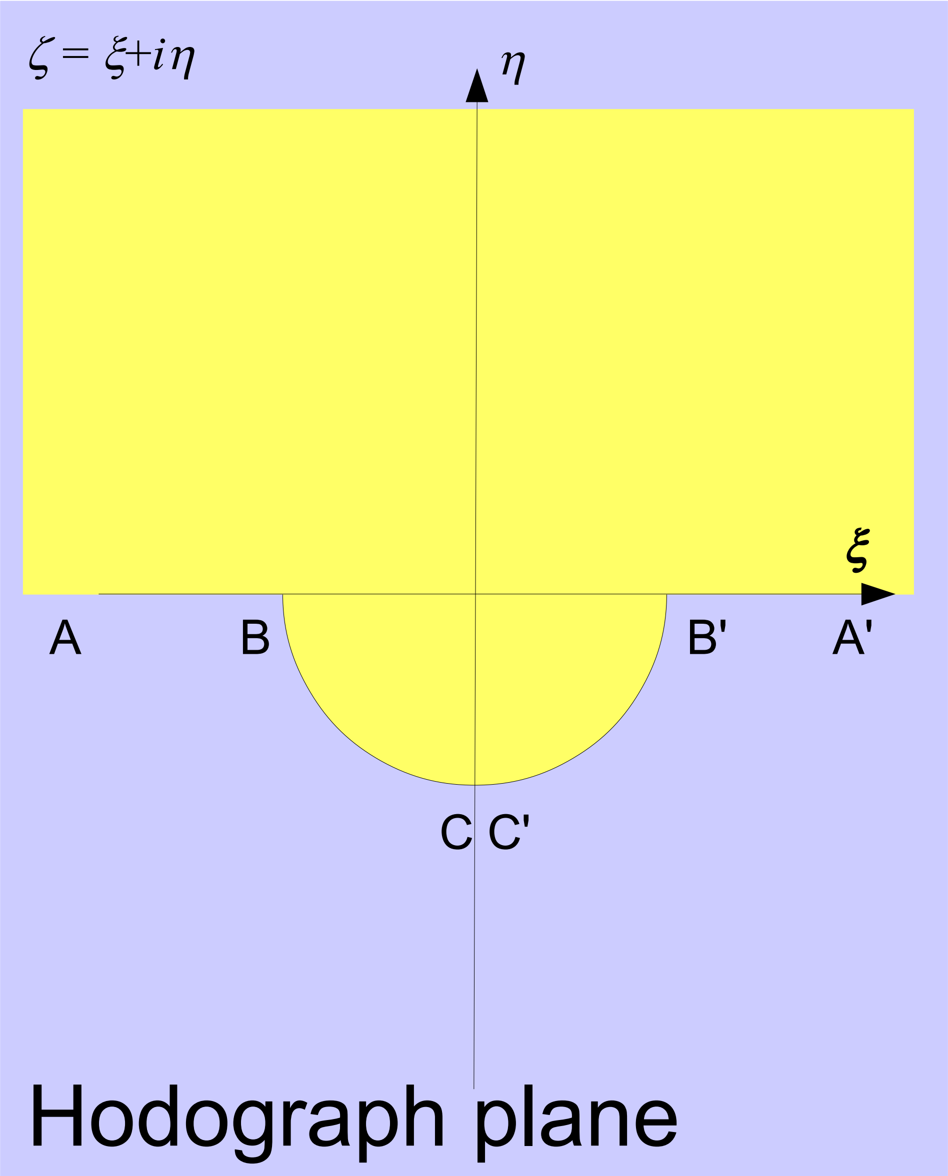
Consider

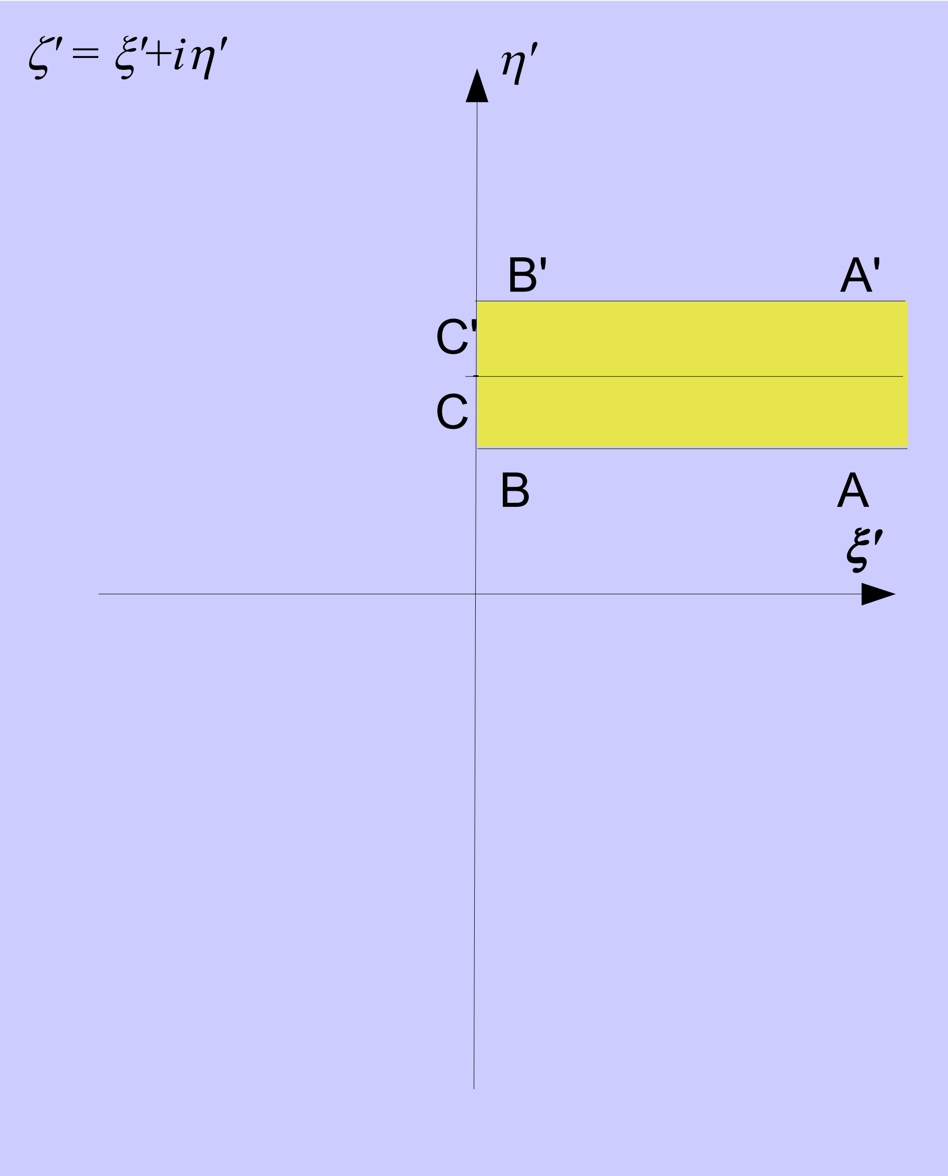
$$\zeta' = \log \zeta$$

$$\zeta' = \log r + i\nu$$

Bottom half of unit circle in ζ-plane maps into $[\pi, 2\pi]$ segment of the vertical axis of the ζ' plane, and...

Know how to map this
to an upper half-plane!





Consider the mapping function from section 4.20

$$\zeta'' = \cosh(\zeta' - i\pi) = -\cosh(\zeta')$$

Shifts rectangle

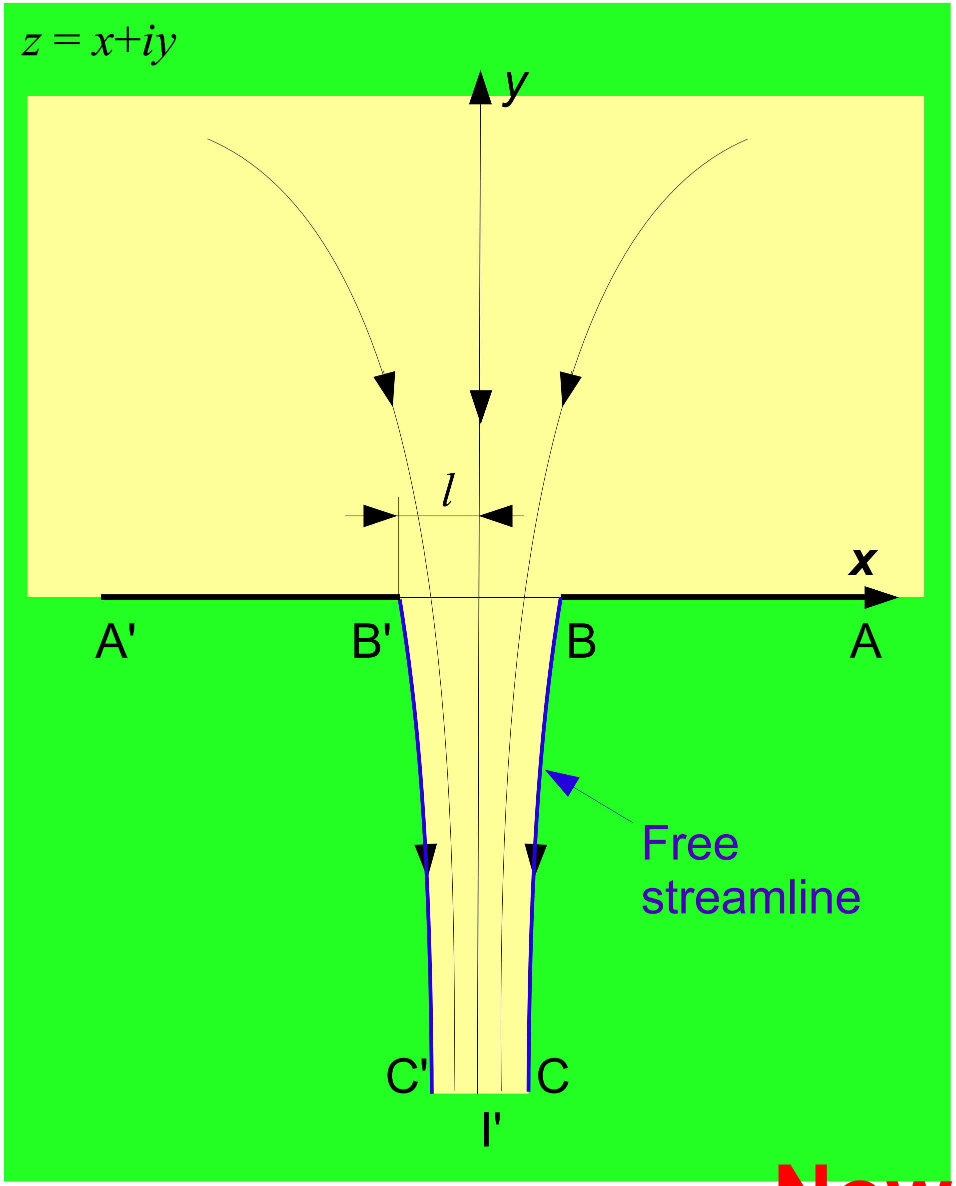
Put a sink at $\zeta'' = 0$

Make the streamline $\psi = 0$ the vertical axis in the original flow

Make $\varphi = 0$ across the hole in the original flow

Also need to determine the shape of the free streamline

Far downstream...



Velocity along the free streamline is U (straight down)

Velocity profile is flat

Let |C'C| = C_c |B'B| = = $2C_c l$

Contraction coefficient

Flow rate between C' and C is $2UC_c l$

$\psi_{I'}$ = 0 (we set it to zero: symmetry...)

Now let's recall something...

Properties of streamfunction

- Streamlines are lines of ψ = *const*
- Difference in the value of ψ between two streamlines equals the volume of fluid flowing between them
- Streamlines ψ = *const* and potential lines φ = *const* are orthogonal at every point in the flow

Discharge rate between I' and C'...

$$\psi_{I'} - \psi_{B'C'} = U\, l\, C_c$$

Discharge rate between I' and C...

$$\psi_{BC} - \psi_{I'} = U\, l\, C_c$$

Now let's recover the complex potential in z-plane

$$F(\zeta'') = -\frac{m}{2\pi} \log \zeta'' + K$$

$$\zeta'' = \cosh(\zeta' - i\pi)$$

$$F(\zeta') = -\frac{m}{2\pi} \log\left(\cosh(\zeta' - i\pi)\right) + K$$

$$F(\zeta')=-\frac{m}{2\pi}\log\left(\cosh(\zeta'-i\pi)\right)+K$$

$$\zeta'=\log\zeta$$

$$F(\zeta)=-\frac{m}{2\pi}\log\left(\cosh(\log\zeta-i\pi)\right)+K$$

$$\zeta=U\frac{dz}{dF}$$

$$F(z)=-\frac{m}{2\pi}\log\left(\cosh\left(\log\left(U\frac{dz}{dF}\right)-i\pi\right)\right)+K$$

Use the set values at B ($\varphi = 0$, $\psi = C_c lU$) to find m, K...

$$F(z) = -\frac{2C_c l U}{\pi} \log\left(\cosh\left(\log\left(U\frac{dz}{dF}\right) - i\pi\right)\right) + iC_c l U$$

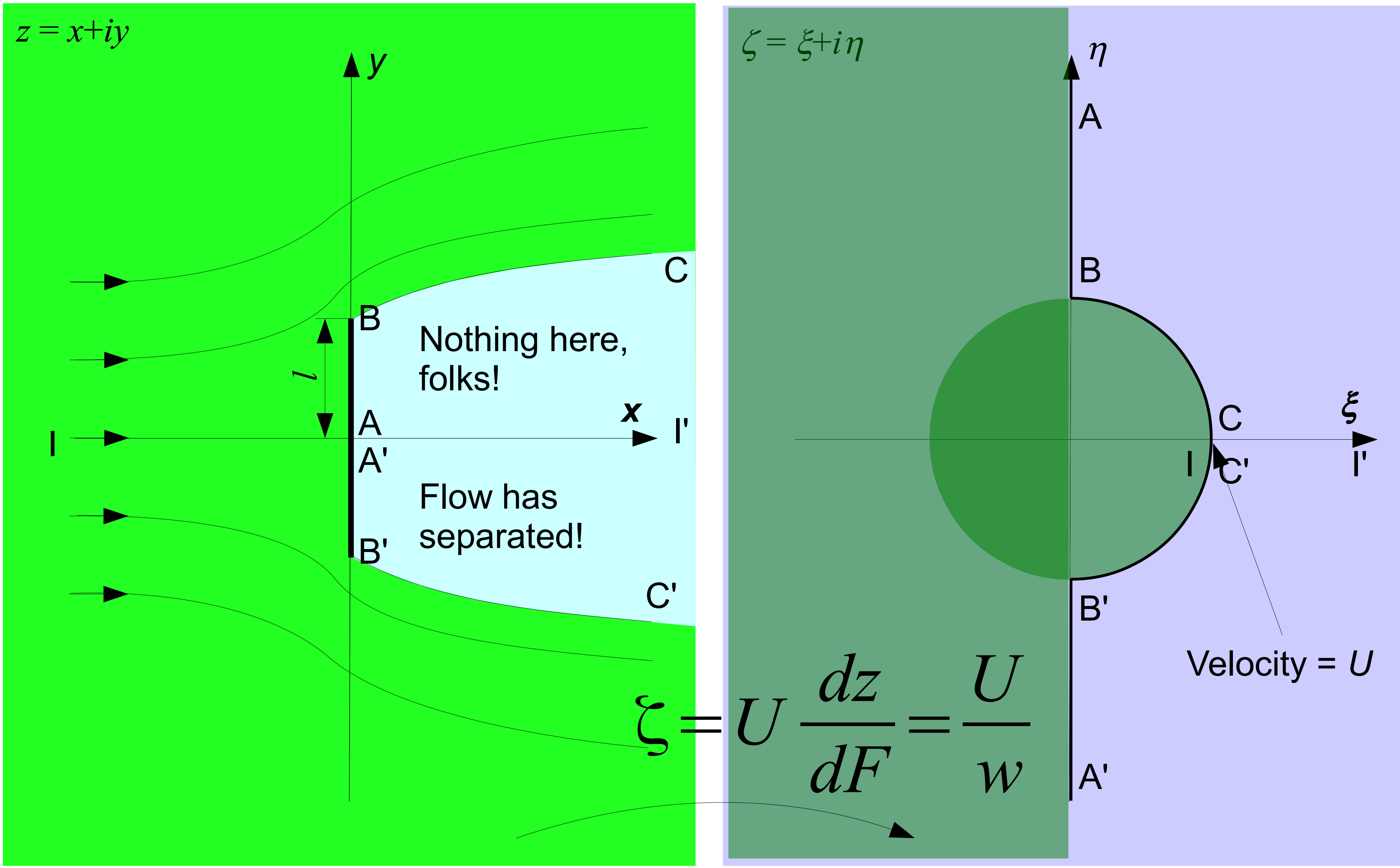
Implicit ODE for $F(z)$ – solve numerically

Simple expression for free streamlines – find C_c:

$$C_c = \frac{\pi}{\pi + 2} \approx 0.611$$

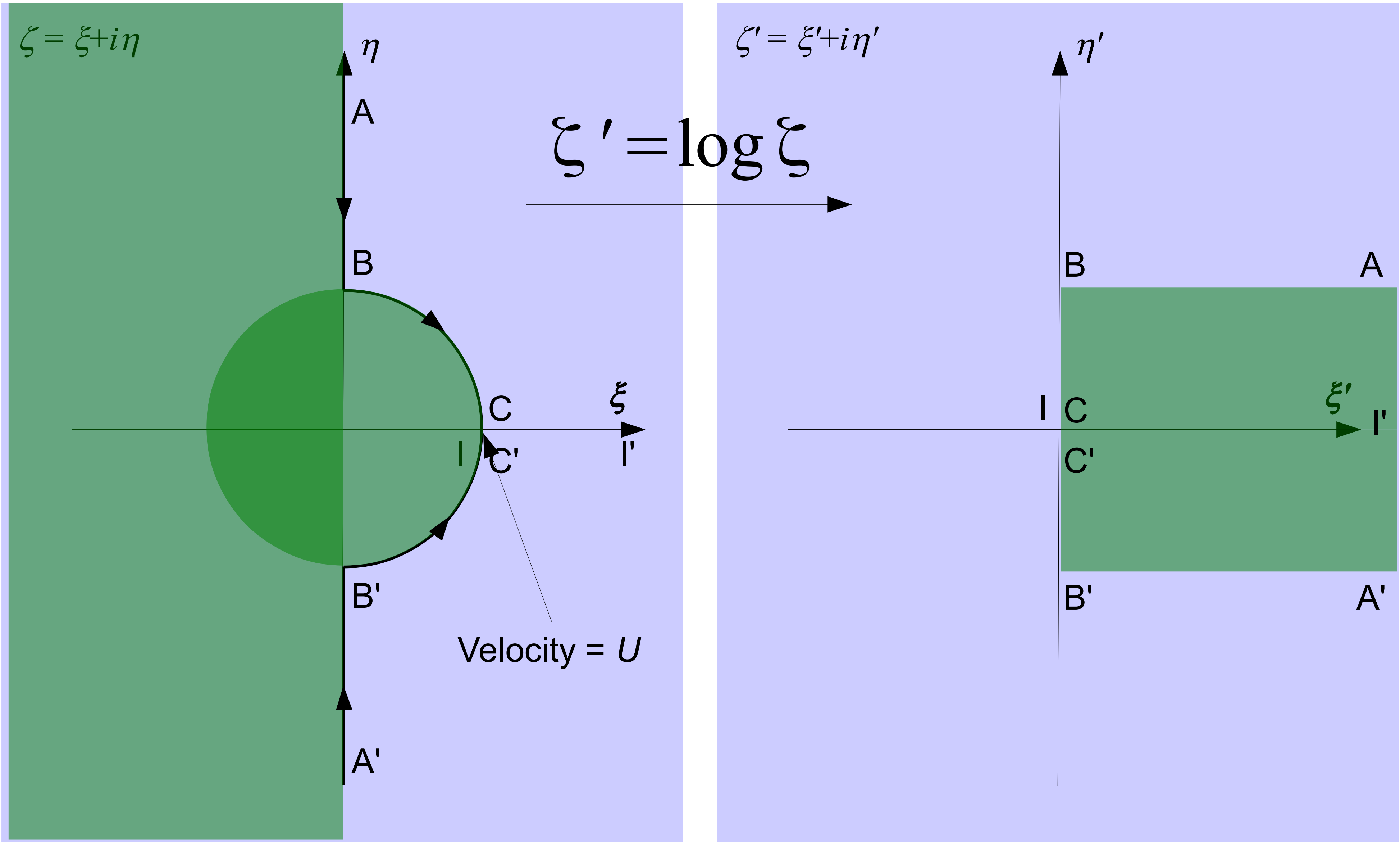
Very close to experimental value for stream out of a slit (despite no gravity, viscosity, real flow not being 2D, *etc*.!)

4.22. Flow past a vertical flat plate

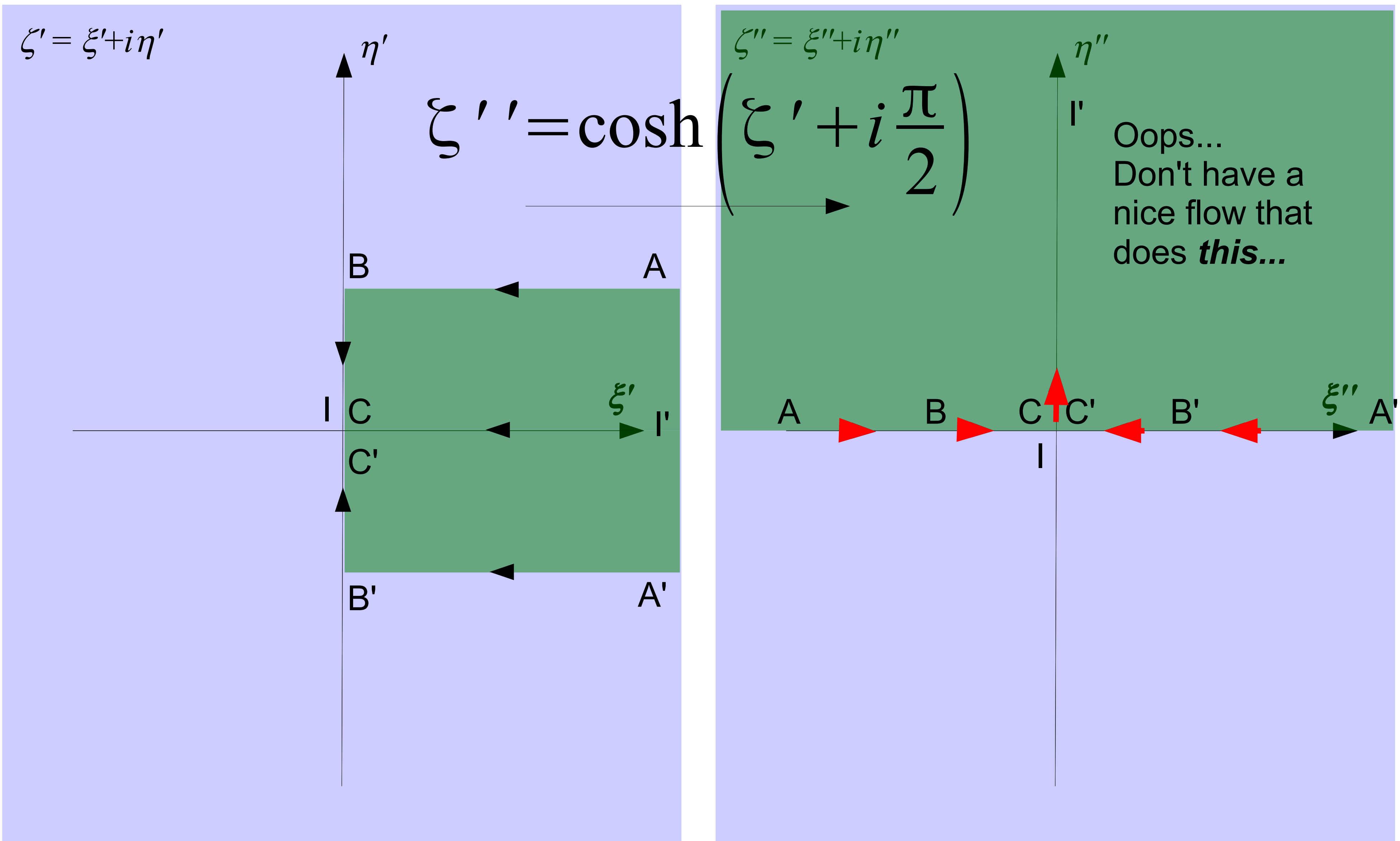


$$\zeta = U\,\frac{dz}{dF} = \frac{U}{w}$$

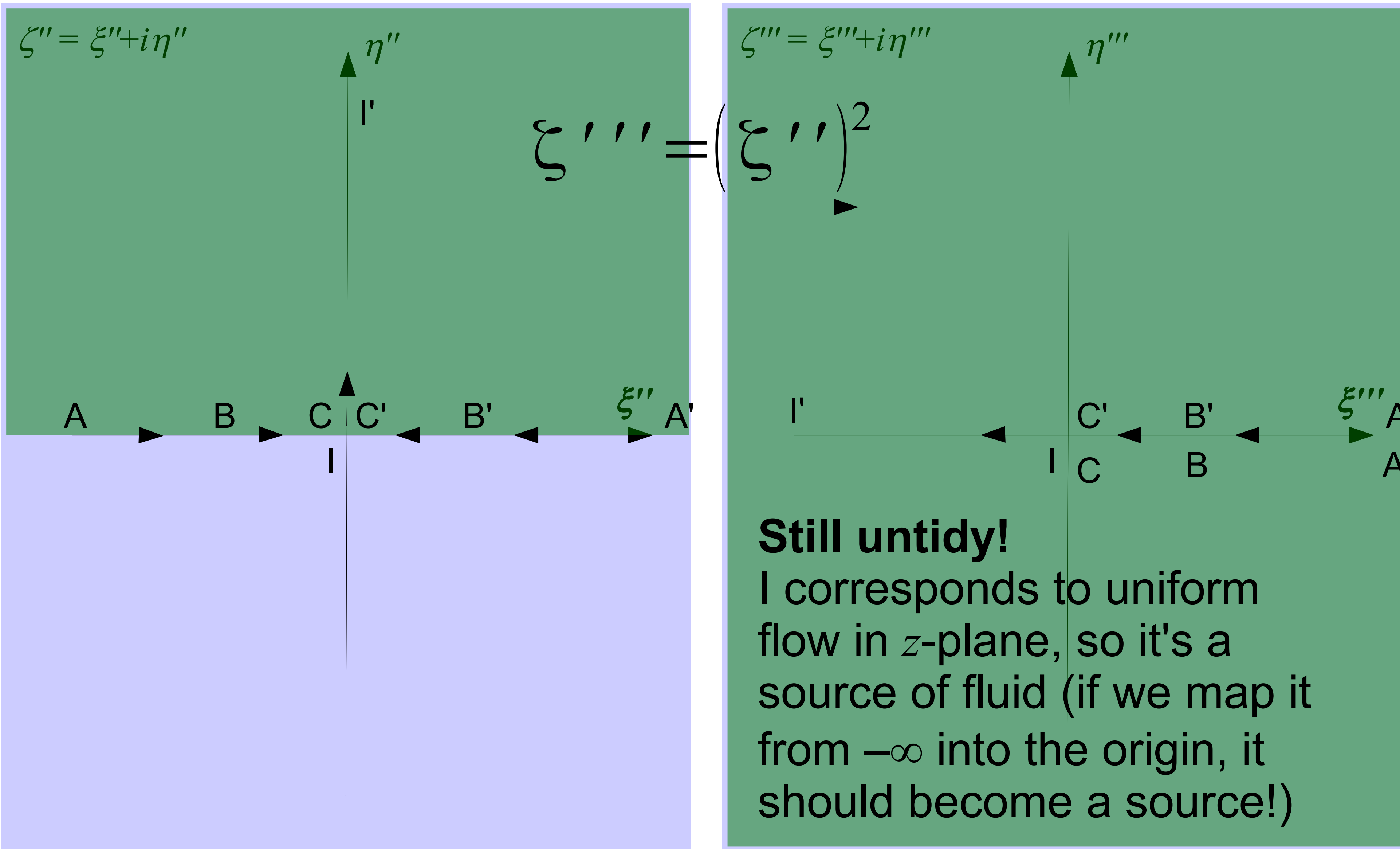
Already know how to map this into a strip...



Also done that...

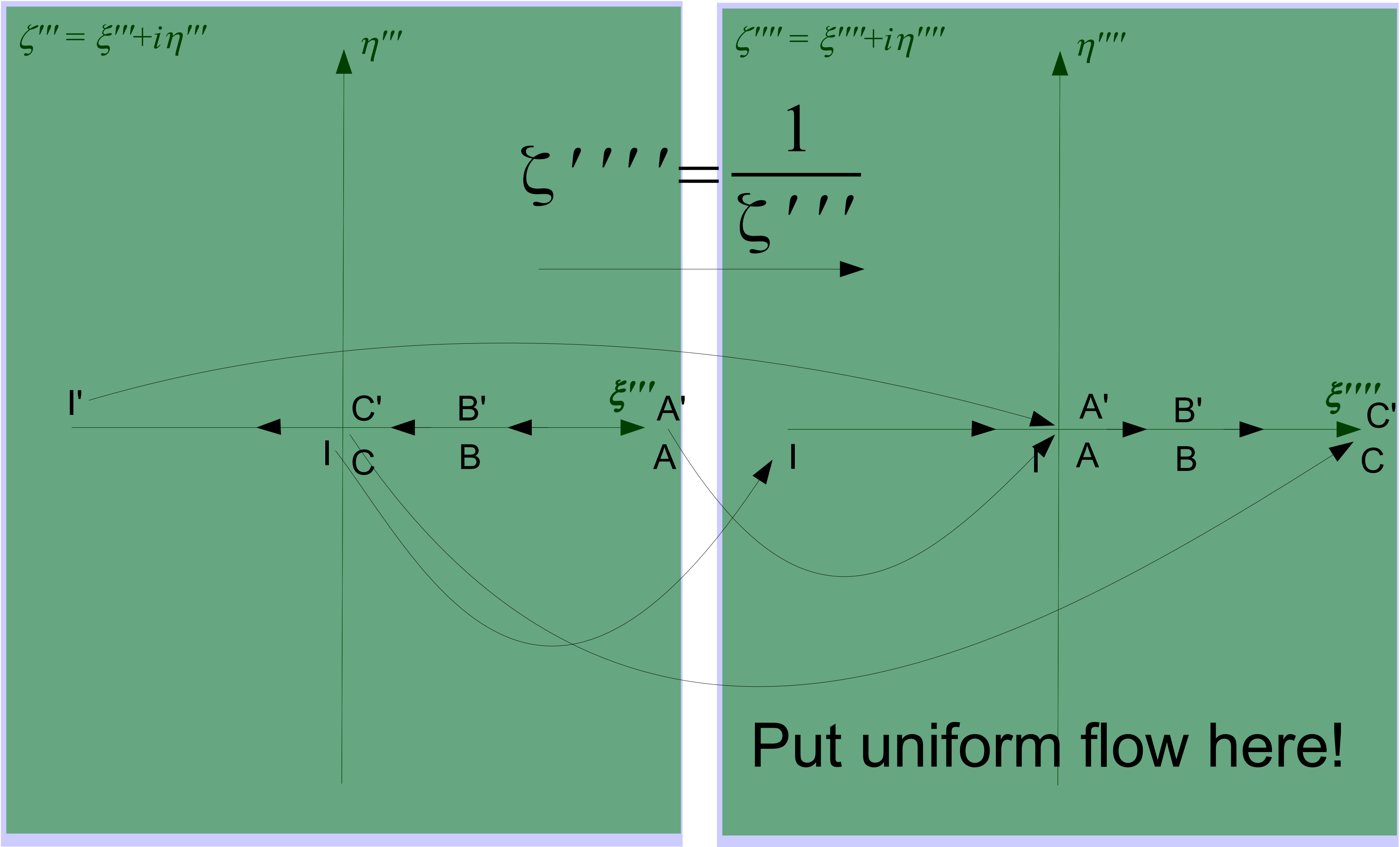


Let's straighten things out...



Still untidy!
I corresponds to uniform flow in z-plane, so it's a source of fluid (if we map it from $-\infty$ into the origin, it should become a source!)

Let's move that pesky source far, far away...



Results

$$F(z) = -\frac{2Ul}{\pi+4}\,\frac{1}{\sinh^2\left[\log\left(U\dfrac{dz}{dF}\right)\right]}$$

ζ'''' ζ''' ζ'' ζ' ζ

Scaling to make velocity = U at $z \to -\infty$

Drag force on the plate (yes, drag force!)

$$X = \frac{2\pi}{\pi+4}\rho U^2 l$$

Separated ideal flows can have drag!