II. Ideal – fluid flow

- Ideal fluids are
  - Inviscid
  - Incompressible
  - The only ones decently understood mathematically
- Governing equations

$$\nabla \cdot \boldsymbol{u} = 0$$

Continuity

$$\frac{\partial \boldsymbol{u}}{\partial t} + \left(\boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \boldsymbol{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \boldsymbol{f}$$

Euler

Boundary conditions

Normal to surface

$$\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{U} \cdot \boldsymbol{n}$$

Velocity of surface

Free-slip
(velocity is parallel to surface)

Potential flow (special case)

$$\boldsymbol{u} = \nabla\varphi \quad (u = \partial\varphi/\partial x,\ v = \partial\varphi/\partial y,\ w = \partial\varphi/\partial z)$$

Potential flow is irrotational

Continuity equation for potential flow

$$\nabla^2\varphi = 0$$

Continuity equation (with boundary conditions) can be solved **alone** for velocity

Then plug φ into momentum equation (Bernoulli form) to solve for pressure

4. 2D potential flows

4.1. Stream function

- 2D ideal continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

- Velocity potential φ

$$u=\frac{\partial \varphi}{\partial x}, \quad v=\frac{\partial \varphi}{\partial y}$$

- Introduce **streamfunction** ψ (counterpart of potential) so that

$$u=\frac{\partial \psi}{\partial y}, \quad v=-\frac{\partial \psi}{\partial x}$$

Streamfunction satisfies continuity equation **by construction**

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Streamfunction exists for any ideal 2D flow

Before going further, consider vorticity in 2D flow

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{bmatrix}$$

Streamfunction satisfies continuity equation **by construction**

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$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ u & v & 0 \end{bmatrix}$$

Vorticity in 2D flow

For 2D, effectively a scalar

$$\boldsymbol{\omega}=\boldsymbol{k}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\boldsymbol{k}\,\omega$$

Now consider an irrotational 2D flow

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$$

Express velocity in terms of streamfunction

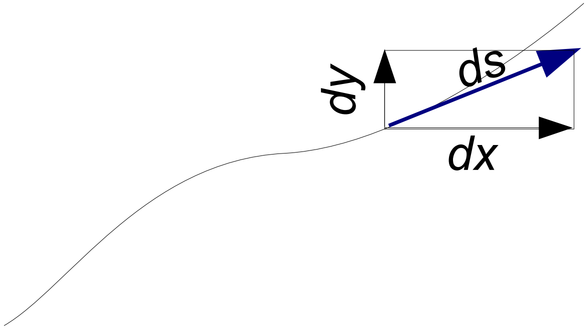
$$\omega=\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)=0$$

$$\nabla^2 \psi=0$$

Properties of streamfunction

- Streamlines are lines of ψ = *const*
- Difference in the value of ψ between two streamlines equals the volume of fluid flowing between them
- Streamlines ψ = *const* and potential lines φ = *const* are orthogonal at every point in the flow

Why ψ = *const* is a streamline

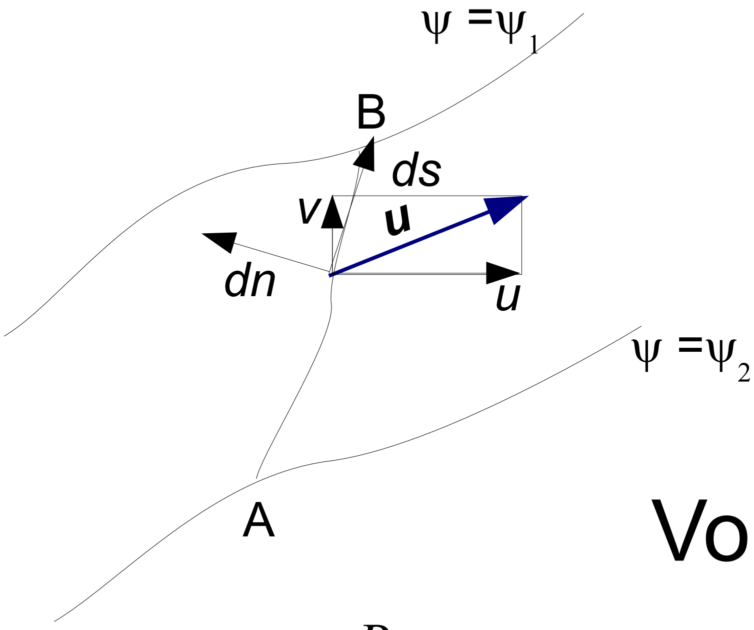


$$d\psi = \frac{d\psi}{ds}ds = \left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial s}\right)ds = -v\,dx + u\,dy$$

$d\psi = 0$ means $v\,dx = u\,dy\,;$ $\boxed{\frac{dy}{v} = \frac{dx}{u}}$

Streamline equation!

Flow rate between two streamlines



Direction along AB:
ds = (*dx*,*dy*)

Direction normal to AB:
dn = (*dy*,-*dx*)

Volume flow rate

$$Q=\int_A^B \boldsymbol{u}\cdot\boldsymbol{n}\,ds=\int_A^B \boldsymbol{u}\cdot d\,\boldsymbol{n}=\int_A^B u\,dy-\int_A^B v\,dx$$

$$Q=\int_A^B d\,\psi=\psi_1-\psi_2$$

Orthogonality between streamlines and potential lines

Along a streamline $d\psi = -v\,dx + u\,dy = 0$

Along an isopotential line (φ = *const*)...

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = u\,dx + v\,dy = 0$$

Normal to streamline: (-*v*, *u*)

Normal to isopotential line: (*u*, *v*)

They are orthogonal: (-*v*, *u*)·(*u*, v) ≡ 0

4.2. Complex potential and velocity

- Complex variable $z = x+iy$
- Function of a complex variable

$$F(z) = \varphi(x,y) + i\ \psi(x,y)$$

- Cauchy-Riemann condition for function of a complex variable to be holomorphic*

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}; \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x};$$

Holomorphic function – complex-valued function of a complex variable which is differentiable in a neighborhood of every point within its domain

Complex potential constructed from velocity potential and streamfunction

$$\mathrm{F}(z) = \varphi\,(x,y) + i\ \psi\,(x,y)$$

Cauchy-Riemann condition satisfied by construction

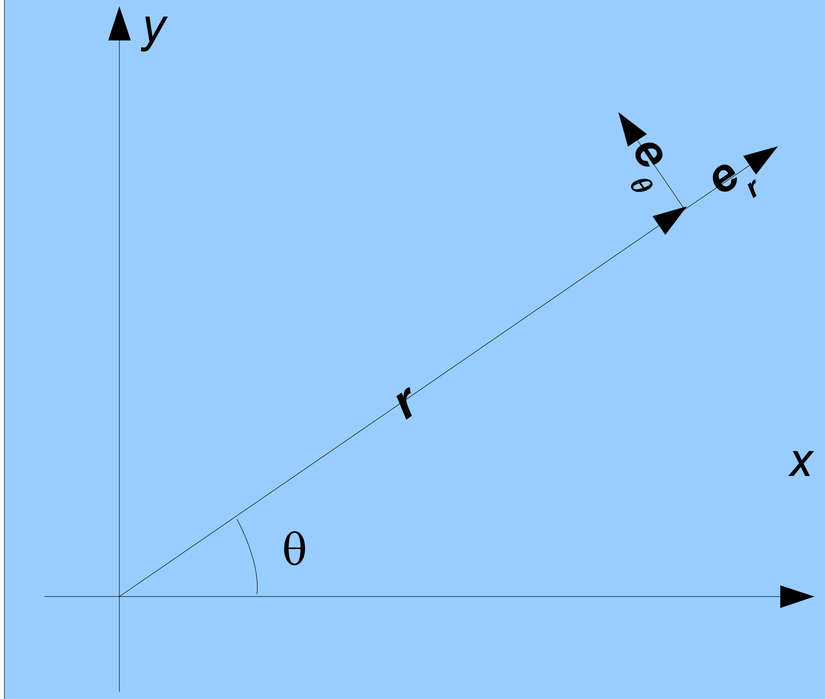
Advantages of using complex potential

- If φ and ψ are the real and imaginary parts of any holomorphic function, $\nabla^2\varphi = 0$ and $\nabla^2\psi = 0$ automatically
- Complex velocity $w = dF/dz = u - iv$ – directly related to flow velocity

Magnitude of complex velocity

$$w^*w = (u + iv)(u - iv) = u^2+v^2 = \boldsymbol{u}\cdot\boldsymbol{u} = \nabla\varphi\cdot\nabla\varphi$$

Polar coordinates in complex plane



$$x + iy = r\,(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$u = u_r\cos\theta - u_\theta\sin\theta$$

$$v = u_r\sin\theta + u_\theta\cos\theta$$

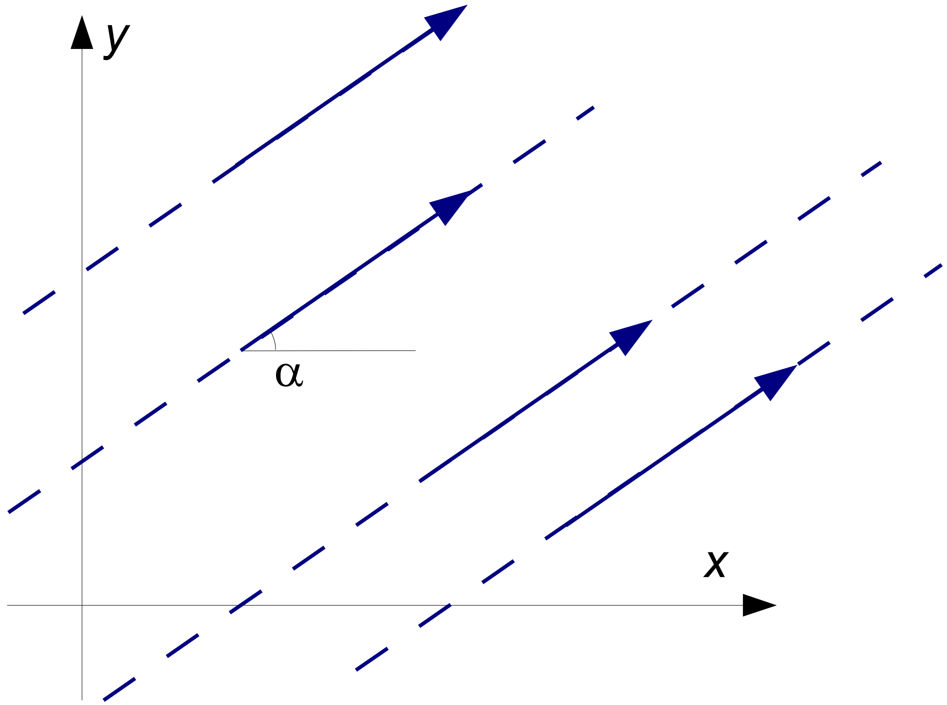
$$w = (u_r - iu_\theta)\,e^{-i\theta}$$

4.3. Uniform flow

$$F(z) = C e^{-i\alpha} z$$

$$w(z) = \frac{dF}{dz} = C e^{-i\alpha} = C \cos\alpha - i C \sin\alpha$$

$$u = C\cos\alpha, \quad v = C\sin\alpha$$

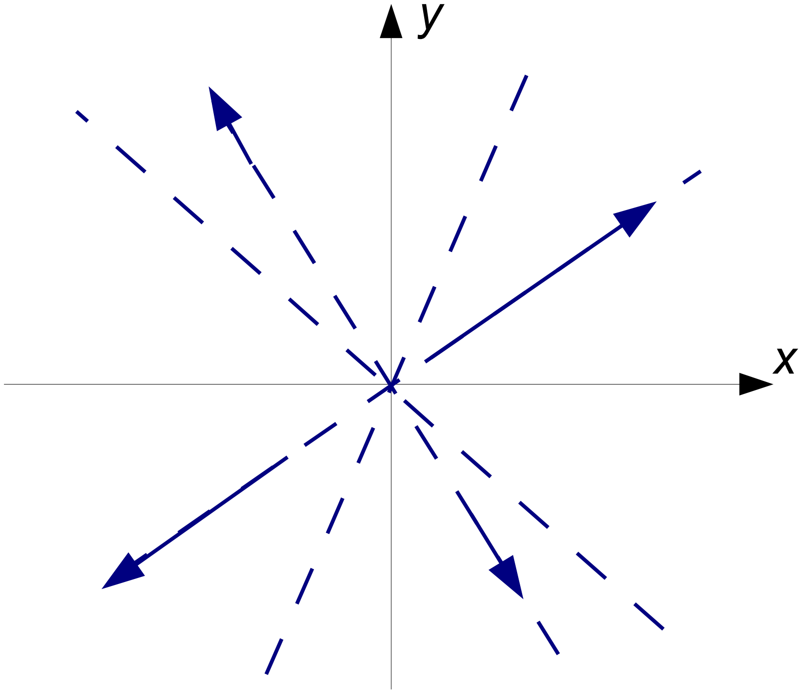


4.4. Source, sink, and vortex

$$F(z)=C\log z=C\log(r\,e^{i\theta})=C(\log r+i\theta)$$

First, let *C* be real and positive

$$\varphi=C\log r,\quad \psi=C\theta$$



$$w(z)=\frac{dF}{dz}=\frac{C}{z}=\frac{C}{r}e^{-i\theta}$$

$$u_r=\frac{C}{r},\quad u_\theta=0$$

Source at $z=0$

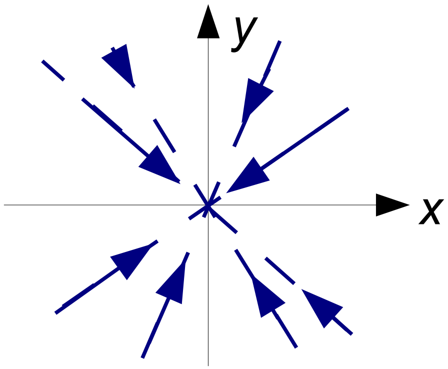
Source strength (discharge rate)

$$m = \int_0^{2\pi} u_r r \, d\theta = \int_0^{2\pi} C \, d\theta = 2\pi C$$

Complex potential of a source of strength m at $z = z_0$

$$F(z) = \frac{m}{2\pi} \log(z - z_0)$$

Complex potential of a sink of strength m at $z = z_0$



$$F(z) = -\frac{m}{2\pi} \log(z - z_0)$$

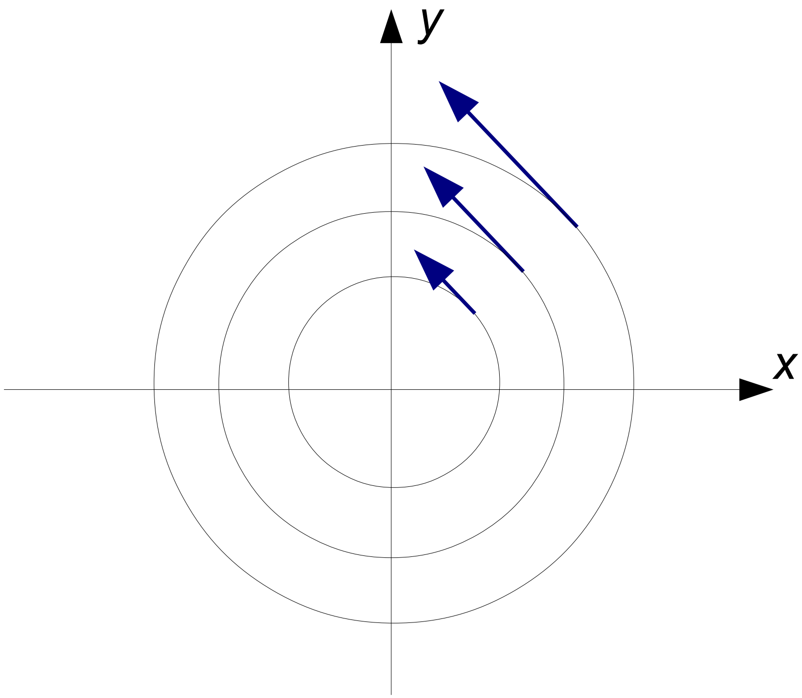
Now consider a purely imaginary constant in the logarithmic potential:

$$F(z) = -iC\log z = -iC\log\left(r e^{i\theta}\right) = -iC\log r + C\theta$$

$$\varphi = C\theta, \quad \psi = -C\log r$$

$$w(z) = \frac{dF}{dz} = -i\frac{C}{z} = -i\frac{C}{r}e^{-i\theta}$$

$$u_r = 0, \quad u_\theta = \frac{C}{r}$$



Point vortex

Vortex strength (circulation)

$$\Gamma = \oint_L \boldsymbol{u} \cdot d\,\boldsymbol{l} = \int_0^{2\pi} u_\theta r\, d\theta = 2\pi C$$

Complex potential of a vortex with circulation Γ at $z = z_0$

$$F(z) = -i\frac{\Gamma}{2\pi}\log(z - z_0)$$

Note 1. $z = z_0$ is a **singularity** ($u_\theta \to \infty$)

Note 2. This flow field is called a **free vortex**:

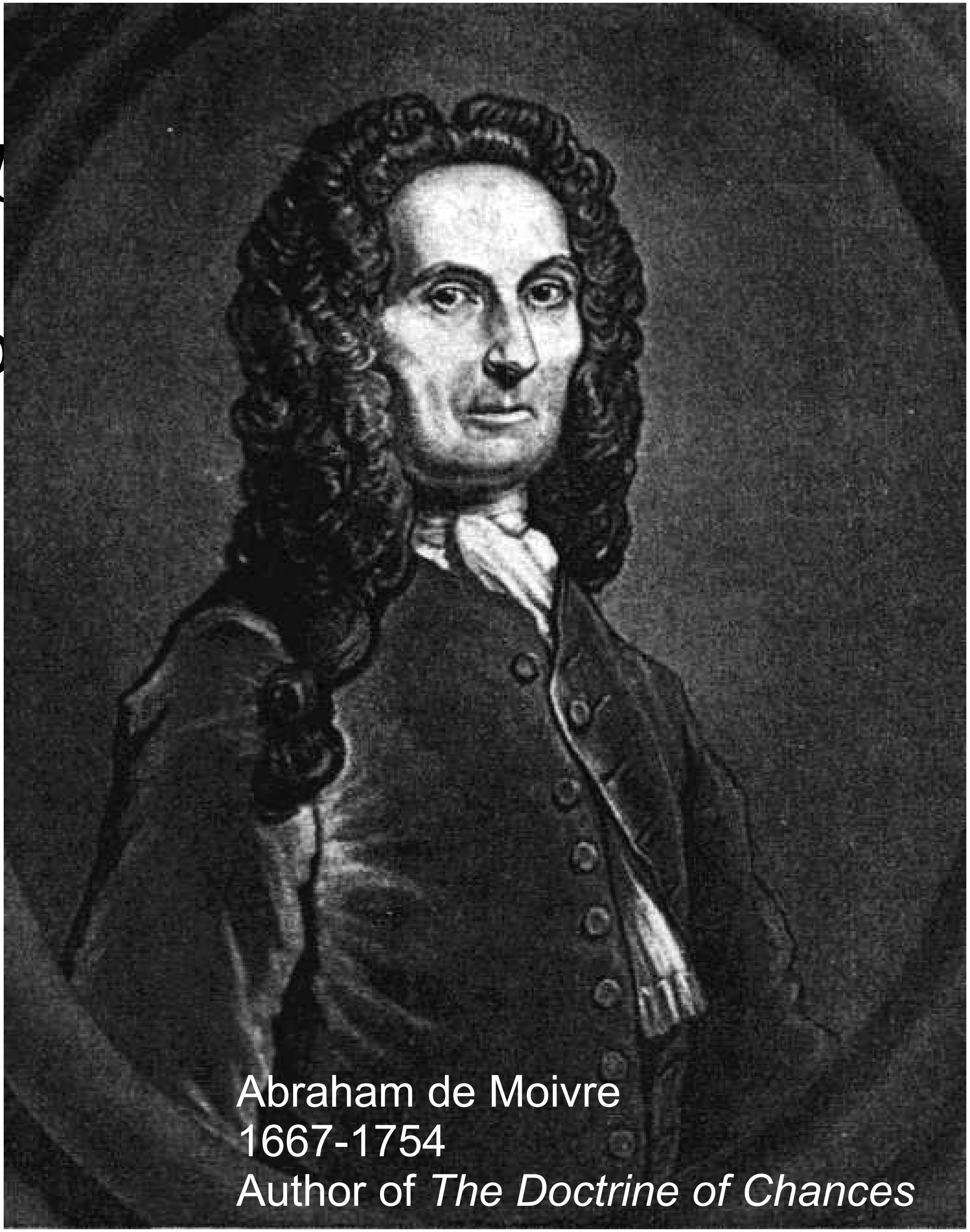
$$\Gamma_{L'} = \oint_{L'} \boldsymbol{u} \cdot d\,\boldsymbol{l} \equiv 0$$

Any contour not including z_0

4.5. Flow in a sector

$$F(z) = U$$

Abraham de Moivre's fo



Abraham de Moivre
1667-1754
Author of *The Doctrine of Chances*

4.5. Flow in a sector

$$F(z)=U\,z^n,\quad n\geqslant 1$$

Abraham de Moivre's formula

$$e^{in\theta}=(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$$

Use polar coordinates

$$z=r\,e^{i\theta}$$

$$F(z)=U\,r^n\cos(n\theta)+i\,U\,r^n\sin(n\theta)$$

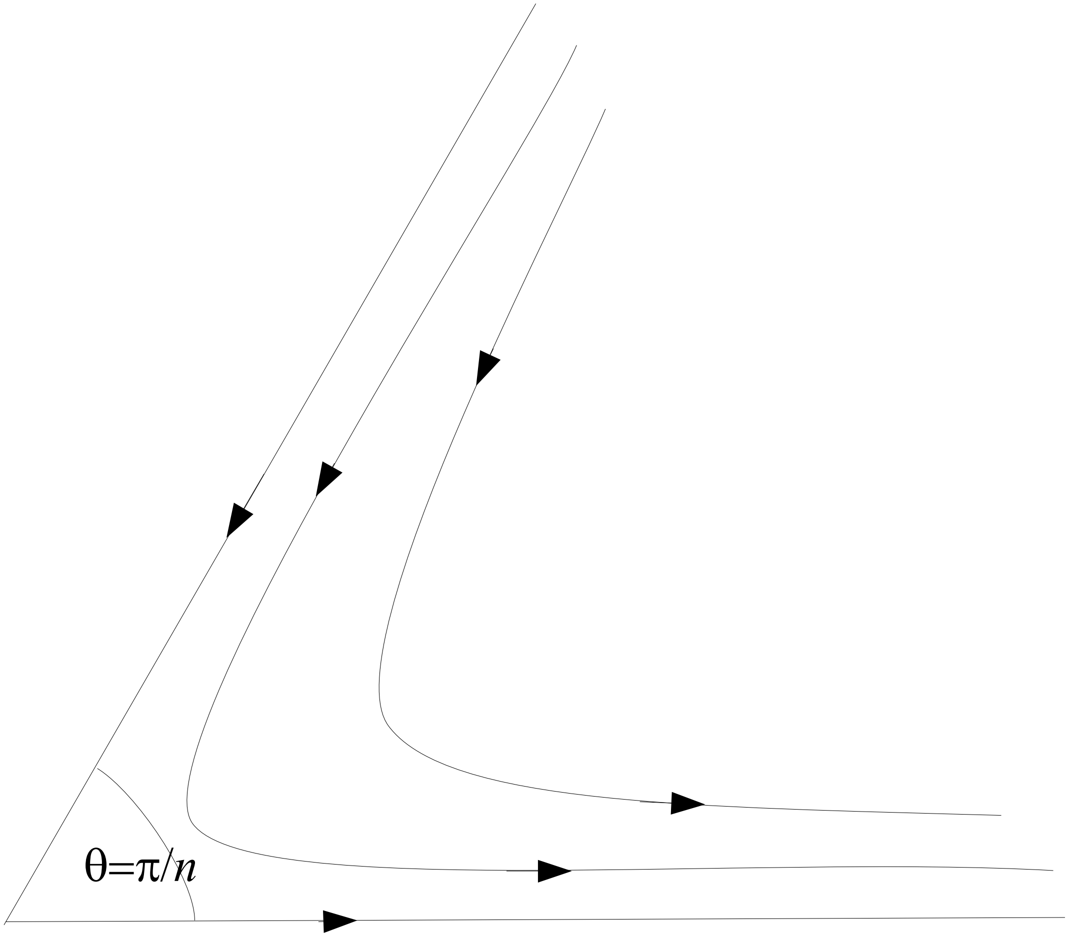
Potential and stream function

$$\varphi=U\,r^n\cos(n\theta),\quad \psi=U\,r^n\sin(n\theta)$$

Complex velocity

$$w(z) = nUz^{n-1} = n\,U\,r^{n-1}e^{i(n-1)\theta} =$$

$$= \left(n\,U\,r^{n-1}\cos n\theta + i\,n\,U\,r^{n-1}\sin n\theta\right)e^{-i\theta}$$



Velocity components

$$u_r = n\,U\,r^{n-1}\cos n\theta$$

$$u_\theta = -n\,U\,r^{n-1}\sin n\theta$$

n = 1: uniform flow

n = 2: flow in a right-angle corner

n = 3: shown

4.6. Flow around a sharp edge

$$F(z)=C z^{1/2}=C r^{1/2} e^{i\theta/2}$$

Potential and streamfunction

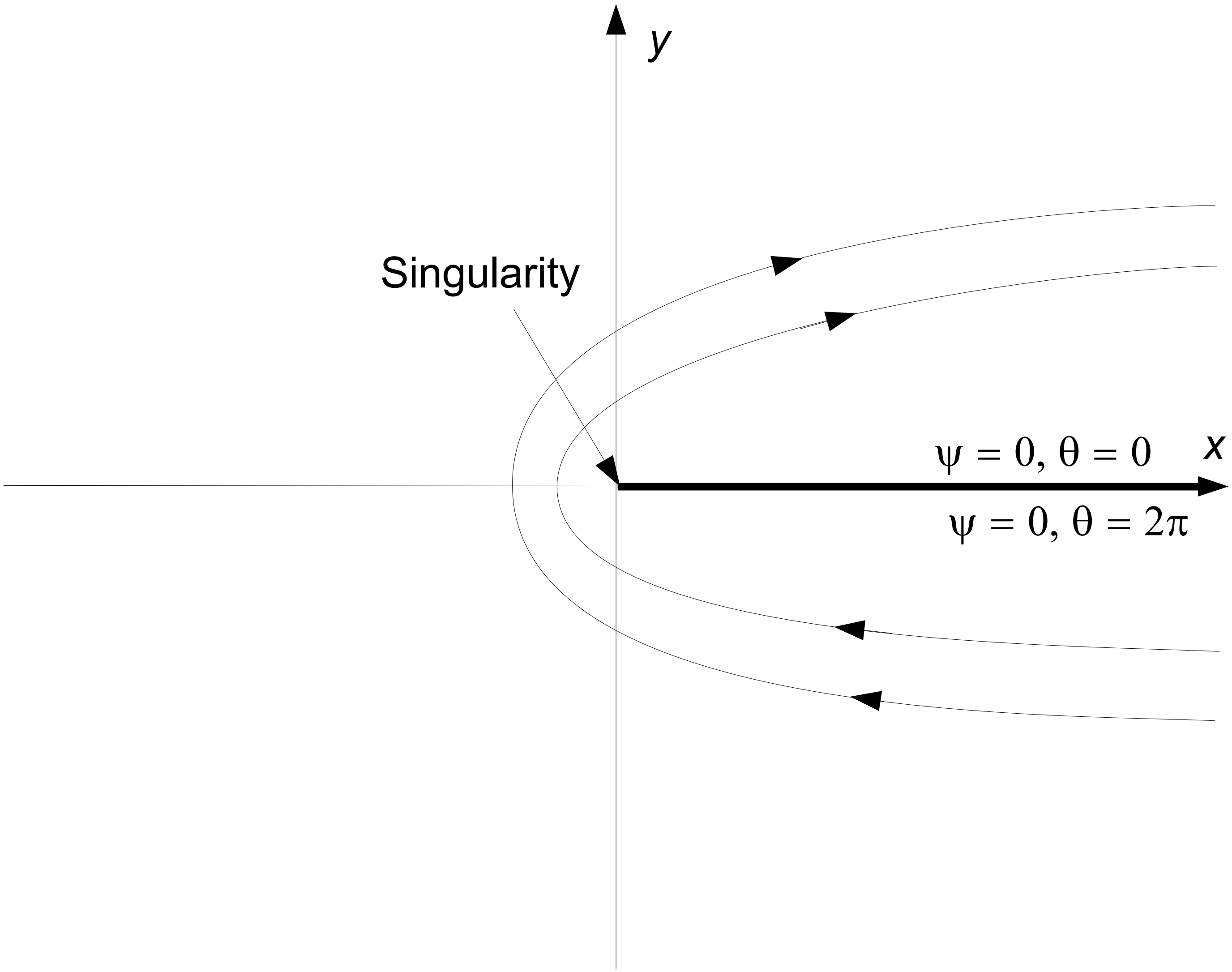
$$\varphi=C r^{1/2}\cos\frac{\theta}{2}, \quad \psi=C r^{1/2}\sin\frac{\theta}{2}$$

Complex velocity

$$w(z)=\frac{dF}{dz}=\frac{1}{2}C z^{-1/2}=\frac{C}{2r^{1/2}}e^{-i\theta/2}=$$

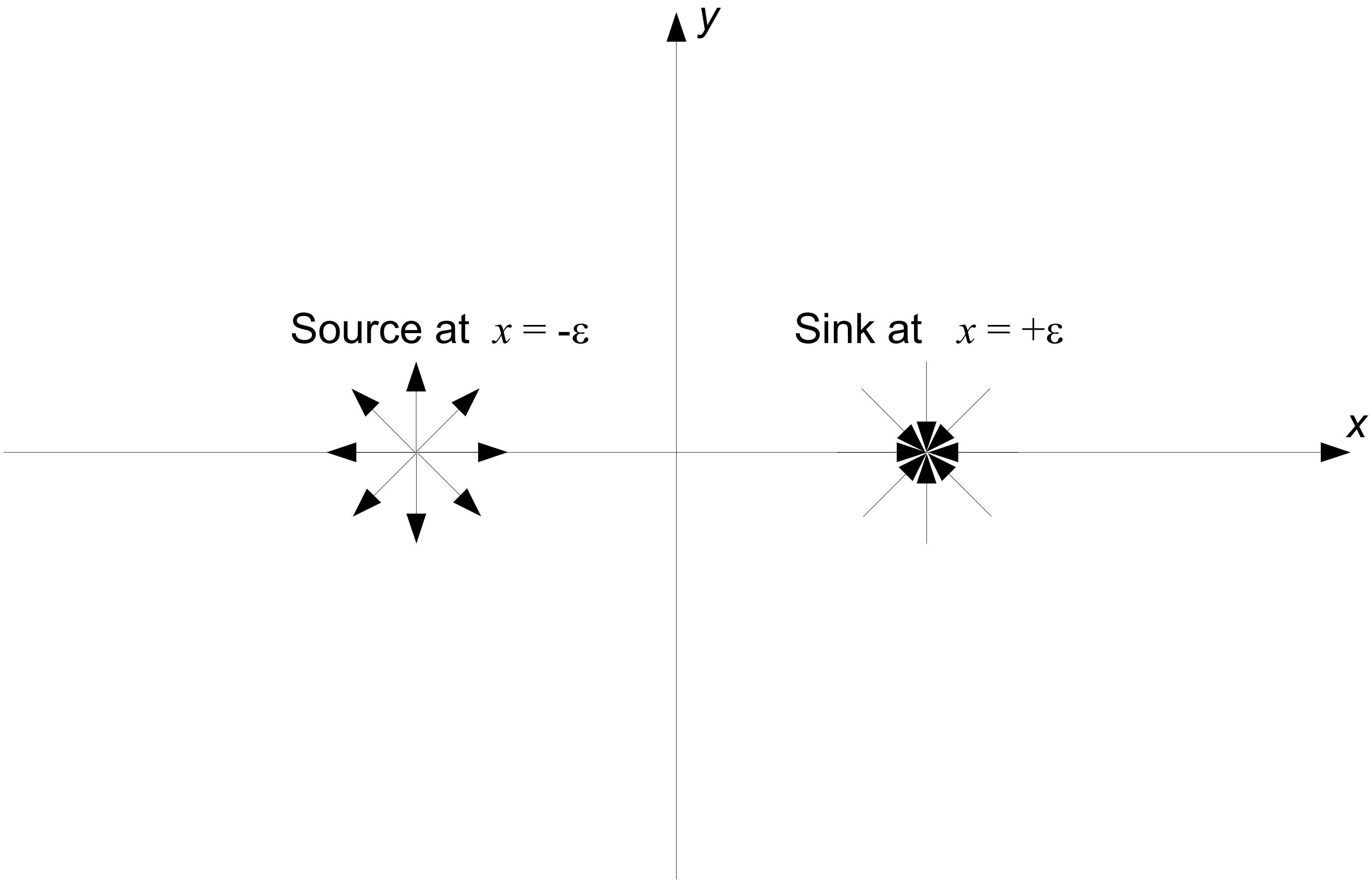
$$=\frac{C}{2r^{1/2}}e^{-i\theta}e^{i\theta/2}=\frac{C}{2r^{1/2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)e^{-i\theta}$$

$$u_r=\frac{C}{2r^{1/2}}\cos\frac{\theta}{2}, \quad u_\theta=-\frac{C}{2r^{1/2}}\sin\frac{\theta}{2}$$



$$u_r = \frac{C}{2r^{1/2}} \cos\frac{\theta}{2}, \quad u_\theta = -\frac{C}{2r^{1/2}} \sin\frac{\theta}{2}$$

4.7. Doublet



Now let $\varepsilon \to 0$

Complex potential of source and sink

$$F(z)=\frac{m}{2\pi}\log(z+\varepsilon)-\frac{m}{2\pi}\log(z-\varepsilon)$$

$$F(z)=\frac{m}{2\pi}\log\frac{z+\varepsilon}{z-\varepsilon}=\frac{m}{2\pi}\log\frac{1+\varepsilon/z}{1-\varepsilon/z}$$

For small ε/z, expand denominator into series:

$$(1-\varepsilon/z)^{-1}=1+\varepsilon/z+\ldots$$

Plug that into $F(z)$

$$F(z)=\frac{m}{2\pi}\log\left(\left(1+\varepsilon/z\right)\left(1+\varepsilon/z+\ldots\right)\right)$$

$$F(z)=\frac{m}{2\pi}\log\left(1+2\frac{\varepsilon}{z}+\ldots\right)$$

Use series expansion for logarithm near 1

$$F(z)=\frac{m}{2\pi}\log\left(1+2\frac{\varepsilon}{z}+\ldots\right)=\frac{m}{2\pi}\left(2\frac{\varepsilon}{z}+\ldots\right)$$

If we take the limit of this as $\varepsilon \to 0$, the result will be trivial: $F(z) = 0$

For a non-trivial result, let $\lim\limits_{\varepsilon\to 0} m\varepsilon=\pi\mu$

Then

$$\lim_{\varepsilon\to 0} F(z)=\frac{\mu}{z}=\frac{\mu}{x+iy}=\mu\frac{x-iy}{(x+iy)(x-iy)}=\mu\frac{x-iy}{x^2+y^2}$$

$$\varphi=\mu\frac{x}{x^2+y^2},\quad \psi=-\mu\frac{y}{x^2+y^2}$$

Consider a streamline $\psi = const$

$$\psi = -\mu \frac{y}{x^2 + y^2}$$

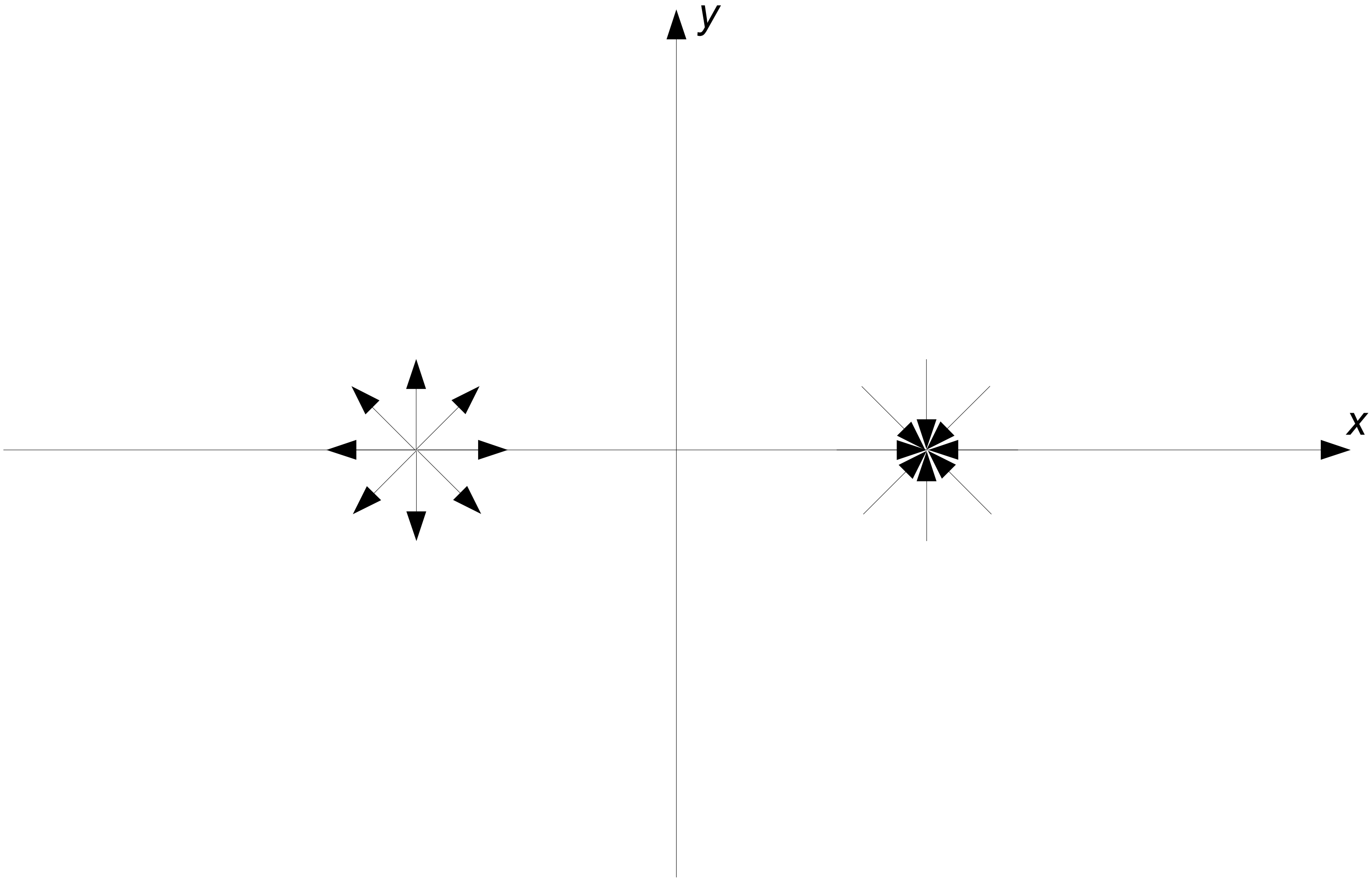
$$\psi\left(x^2 + y^2\right) = -\mu\, y$$

$$x^2 + y^2 + \frac{\mu}{\psi}\, y = 0$$

$$x^2 + y^2 + \frac{\mu}{\psi}\, y + \left(\frac{\mu}{2\,\psi}\right)^2 = \left(\frac{\mu}{2\,\psi}\right)^2$$

$$x^2 + \left(y + \frac{\mu}{2\,\psi}\right)^2 = \left(\frac{\mu}{2\,\psi}\right)^2$$

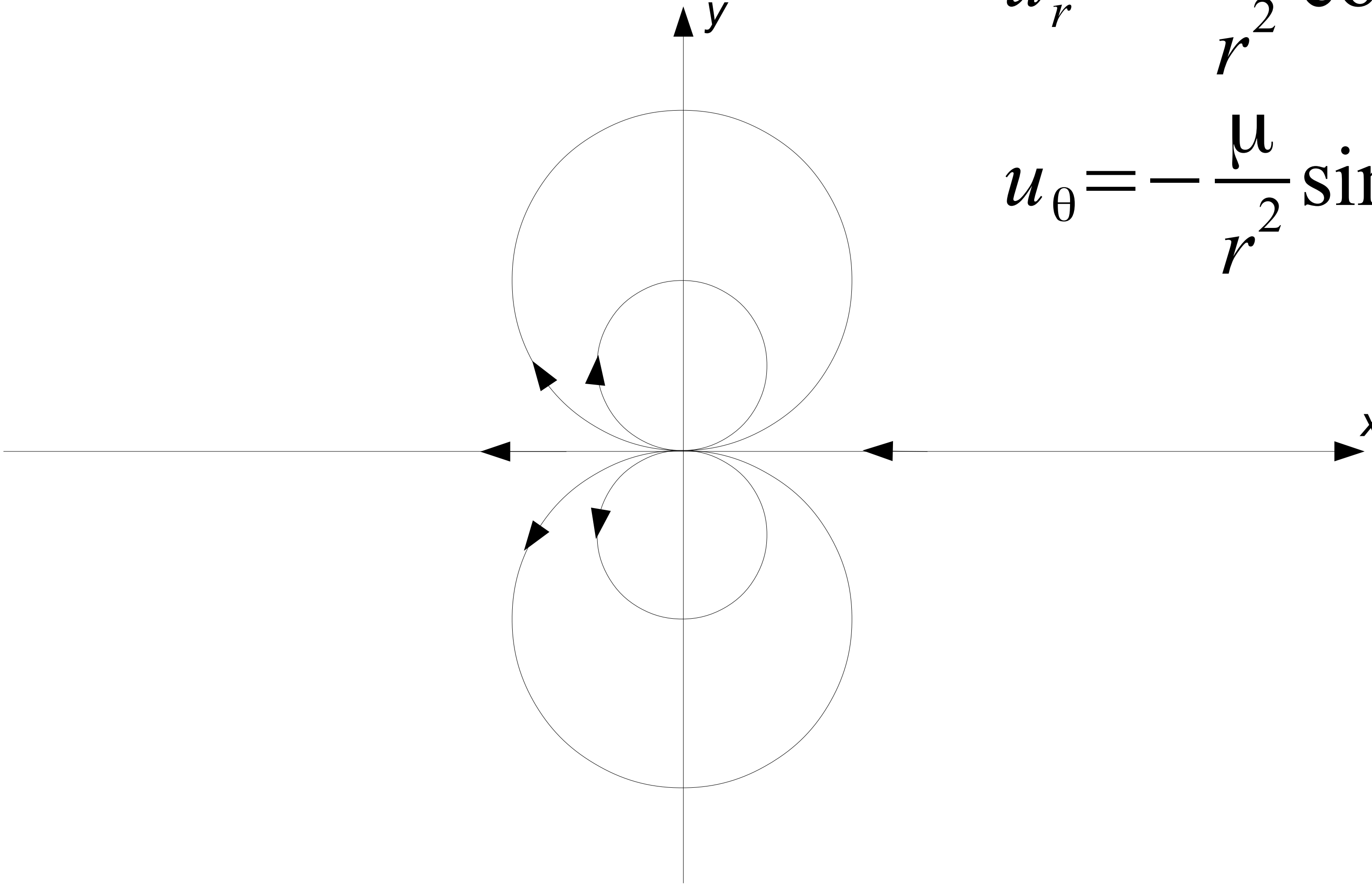
Circle of radius $\mu/(2\psi)$ and center at $x = 0$, $y = -\mu/(2\psi)$


y
x

$$w(z) = -\frac{\mu}{z^2} = -\frac{\mu}{r^2} e^{-2i\theta} = -\frac{\mu}{r^2} e^{-i\theta}(\cos\theta - i\sin\theta)$$

$$u_r = -\frac{\mu}{r^2}\cos\theta$$

$$u_\theta = -\frac{\mu}{r^2}\sin\theta$$



Doublet of strength μ at $z = z_0$

$$F(z) = \frac{\mu}{z - z_0}$$

4.8. Circular cylinder flow

Let uniform flow go past a doublet

$$F(z) = Uz + \frac{\mu}{z}$$

Potential and stream function

$$F(z) = Ure^{i\theta} + \frac{\mu}{r\,e^{i\theta}} = \underset{\text{Potential}}{\left(Ur + \frac{\mu}{r}\right)\cos\theta} + i\underset{\text{Stream function}}{\left(Ur - \frac{\mu}{r}\right)\sin\theta}$$

Consider streamline $\psi = 0$

$Ur = \mu/r$ means that this streamline is a circle of radius $a = (\mu/U)^{1/2}$

Can rewrite complex potential as

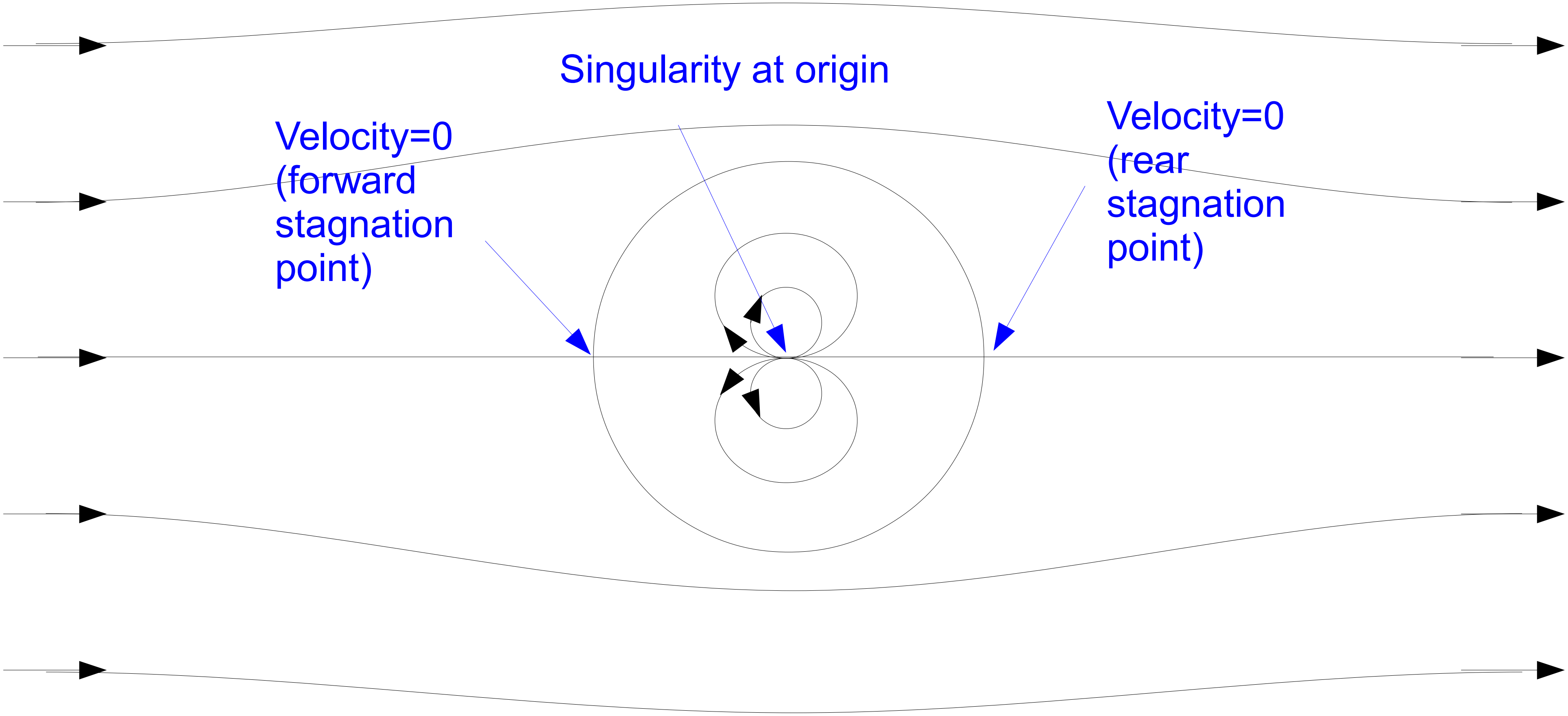
$$F(z) = U\left(z + \frac{a^2}{z}\right)$$

$$z \to \infty, \quad F(z) \to U\,z$$

Uniform flow dominates the far field

$$z \to 0, \quad F(z) \to U\,\frac{a^2}{z}$$

Doublet dominates the flow near the origin



Singularity at origin

Velocity=0
(forward
stagnation
point)

Velocity=0
(rear
stagnation
point)

Flow symmetry: $F(-z) = -F(z)$

4.9. Cylinder with circulation

Take cylinder flow, add rotation around the origin

$$F(z)=U\left(z+\frac{a^2}{z}\right)+\frac{i\Gamma}{2\pi}\log z+C$$

Vortex at origin

Constant to keep $\psi = 0$ at $r = a$

Pretty easy to find C, tuck it into the logarithm

$$F(z)=U\left(z+\frac{a^2}{z}\right)+\frac{i\Gamma}{2\pi}\log\frac{z}{a}$$

Complex velocity

$$w=\frac{dF}{dz}=U\left(1-\frac{a^2}{z^2}\right)+\frac{i\Gamma}{2\pi}\frac{1}{z}$$

$$w = U\left(1 - \frac{a^2}{z^2}\right) + \frac{i\Gamma}{2\pi}\frac{1}{z} = U\left(1 - \frac{a^2}{r^2}e^{-2i\theta}\right) + \frac{i\Gamma}{2\pi}\frac{1}{r}e^{-i\theta}$$

$$w = \left[U\left(e^{i\theta} - \frac{a^2}{r^2}e^{-i\theta}\right) + \frac{i\Gamma}{2\pi}\frac{1}{r}\right]e^{-i\theta}$$

$$w = \left[U\left(1 - \frac{a^2}{r^2}\right)\cos\theta + i\left(U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}\right)\right]e^{-i\theta}$$

Remember that $w = (u_r - iu_\theta)e^{-i\theta}$

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \quad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}$$

On the surface ($r = a$),

$$u_r = 0, \quad u_\theta = -2U\sin\theta - \frac{\Gamma}{2\pi a}$$

Boundary!

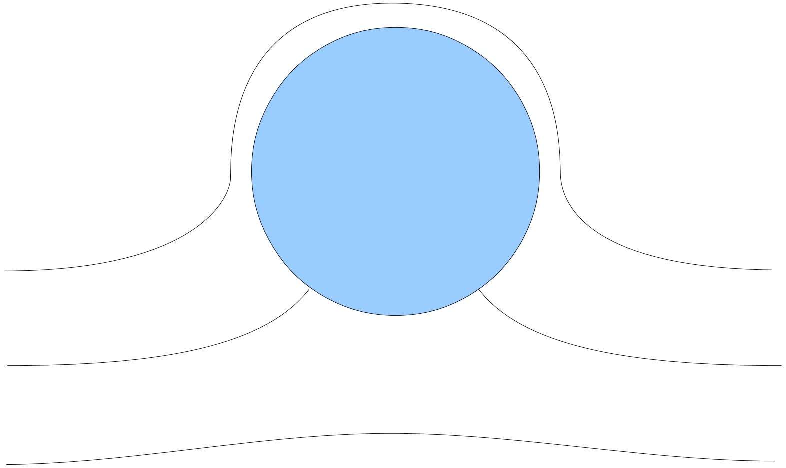
Find stagnation points (velocity = 0, $r = a$)

$$\sin\theta_s = -\frac{\Gamma}{4\pi U a}$$

Possibilities:

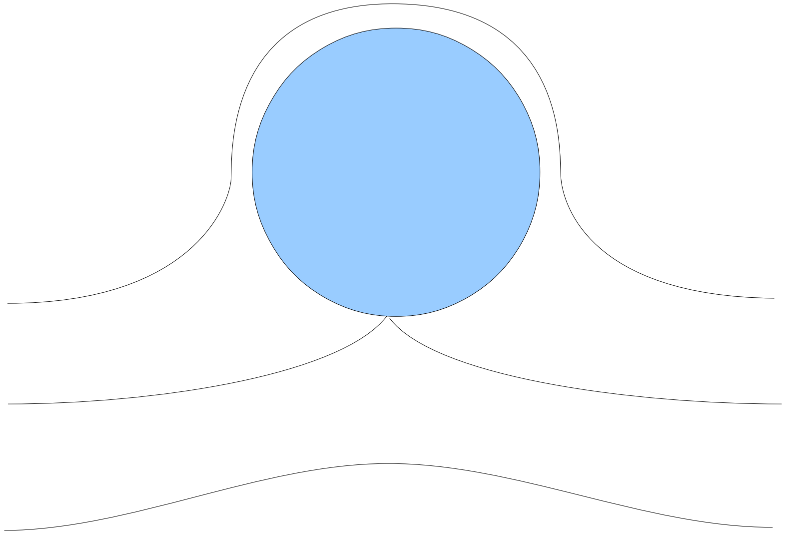
- 2 stagnation points on the cylinder
- 1 stagnation point on the cylinder
- 0 stagnation points on the cylinder (but maybe somewhere else in the flow?)

Two stagnation points



$$0 < \frac{\Gamma}{4\pi Ua} < 1$$

One stagnation point



$$\frac{\Gamma}{4\pi Ua} = 1$$

No stagnation points on the cylinder

$$\frac{\Gamma}{4\pi U a} > 1$$

Look for stagnation point (r_s, θ_s) elsewhere (for $r_s > a$)

Cannot be 0

Must be 0!

$$u_r = U\left(1 - \frac{a^2}{r_s^2}\right)\cos\theta_s = 0,$$

$$u_\theta = -U\left(1 + \frac{a^2}{r_s^2}\right)\sin\theta_s - \frac{\Gamma}{2\pi r_s} = 0$$

$\cos\theta_s = 0$ means $\theta_s = \pi/2$ or $\theta_s = 3\pi/2$

$$\underbrace{U\left(1+\frac{a^2}{r_s^2}\right)}_{\text{positive}}\sin\theta_s = \underbrace{-\frac{\Gamma}{2\pi r_s}}_{\text{negative}}$$

Must be -1, so $\theta_s = 3\pi/2$

$$U\left(1+\frac{a^2}{r_s^2}\right) = \frac{\Gamma}{2\pi r_s}$$

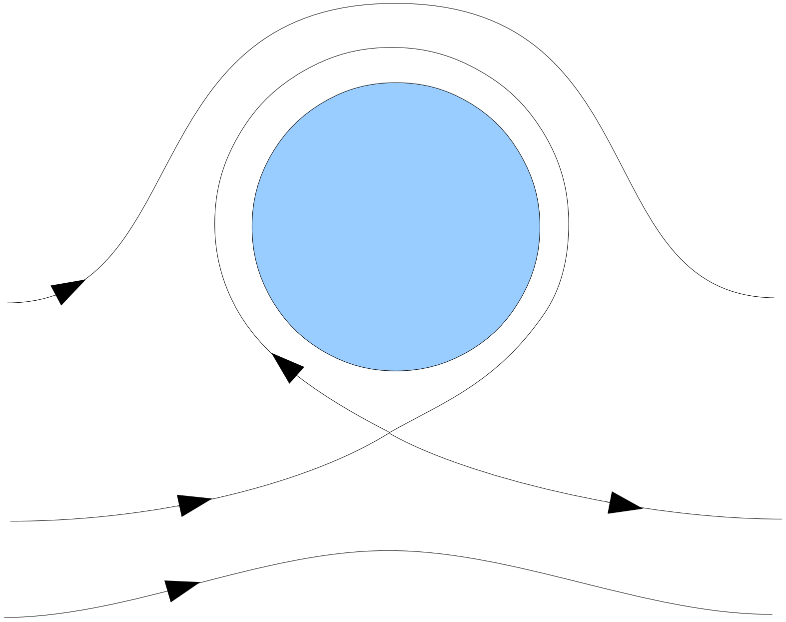
Solve this for r_s

$$r_s = \frac{\Gamma}{4\pi U} \pm \sqrt{\left(\frac{\Gamma}{4\pi U}\right)^2 - a^2}$$

Two stagnation points

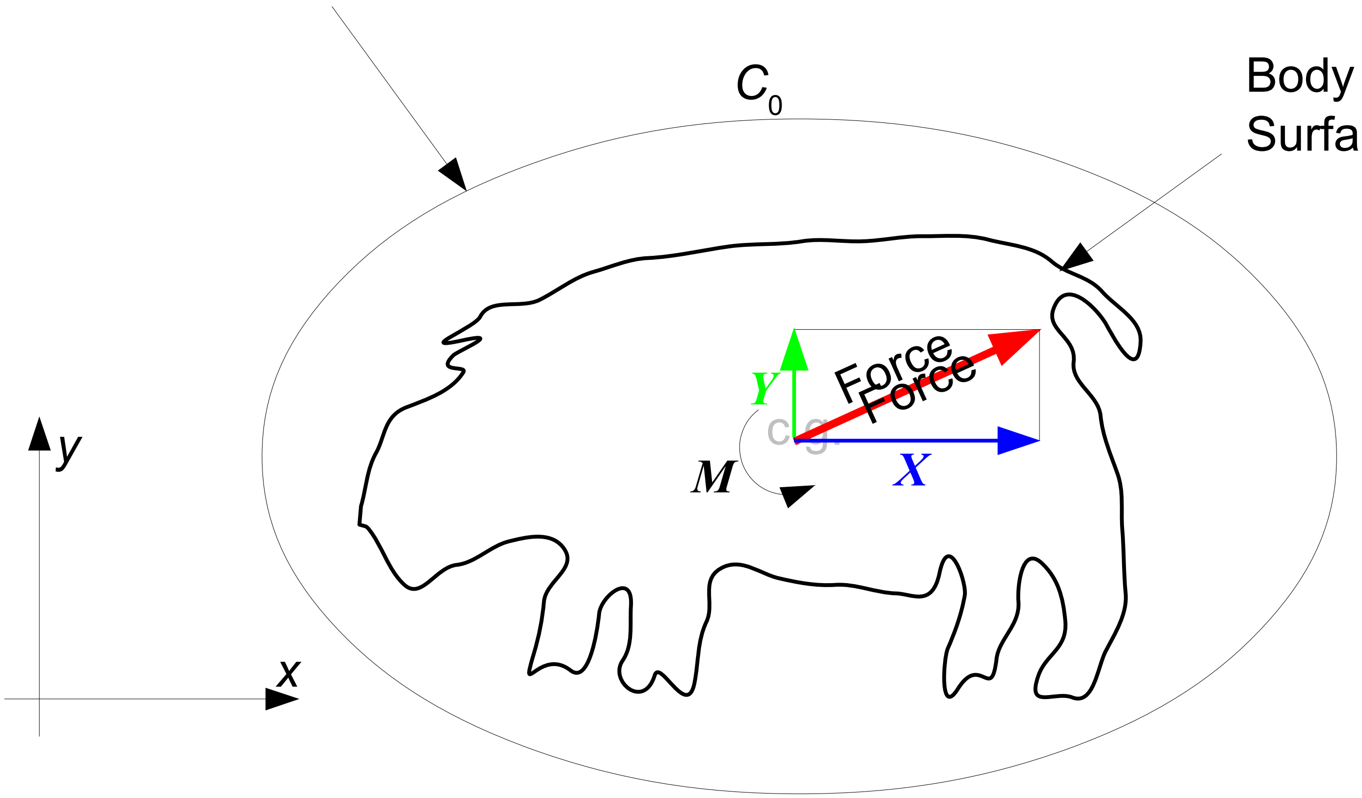
- inside the cylinder (so who cares?)

+ outside the cylinder (good stuff)



4.10. Blasius integral laws

- Find potential
- Find velocity components
- Plug velocity into Bernoulli equation to find pressure on body surface
- Integrate to find
  - Hydrodynamic force on the body
  - Hydrodynamic moment on the body
- **MUCH simpler with complex potential!**



Any contour fully enclosing the body

Body of an arbitrary shape
Surface: streamline $\psi = 0$

Complex force:

$X - iY$

Blasius first law

$$X - iY = i\frac{\rho}{2}\oint_{C_0} w^2\, dz$$

Blasius second law

$$M = \frac{\rho}{2}\Re\left(\oint_{C_0} z\, w^2\, dz\right)$$

Evaluating complex integrals

Taylor series (real variable)

$$f(x-x_0)=\sum_{n=0}^{\infty} a_n (x-x_0)^n, a_n=\frac{f^{(n)}(x_0)}{n!}$$

This expansion is valid in an interval $|x - x_0| < \delta x$

Evaluating complex integrals

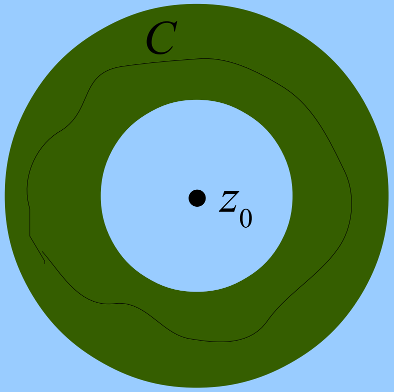
Laurent series (complex variable)

$$f(z-z_0)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

$$a_n=\frac{1}{2\pi i}\oint_C f(\zeta)(\zeta-z_0)^{-n-1}\,d\zeta$$

This expansion is valid in an annulus where f is holomorphic: $R_1 < |z - z_0| < R_2$

If $R_1 = 0$, z_0 – isolated singularity



Coefficient a_{-1} of Laurent series:

residue of f at z_0

Cauchy theorem

If complex function $f(z)$ is holomorphic everywhere inside contour C,

$$\oint_C f(z)\,dz = 0$$

Cauchy residue theorem

If complex function $f(z)$ is holomorphic everywhere inside contour C, except isolated singularities,

$$\oint_C f(z)\,dz = 2\mathrm{i}\pi \sum_k a_{-1,k}$$

Example

e^z – holomorphic everywhere in a disk of radius r with center at $z = 0$

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots$$

e^z/z – holomorphic everywhere in a disk of radius r with center at $z = 0$, except at it center

$$\frac{e^z}{z} = \underbrace{\frac{1}{z}}_{a_{-1}=1} + 1 + \frac{z}{2} + \frac{z^2}{6} + \ldots$$

Note. $a_{-m} \neq 0$, $a_{-m-k} \equiv 0$, $k = 1,2, \ldots$ at $z = z_0$ –

z_0 is a **pole** of order m

4.11. Force and moment on a circular cylinder

Complex potential

$$F(z)=U\left(z+\frac{a^2}{z}\right)+\frac{i\Gamma}{2\pi}\log\frac{z}{a}$$

Complex velocity

$$w=\frac{dF}{dz}=U\left(1-\frac{a^2}{z^2}\right)+\frac{i\Gamma}{2\pi z}$$

Blasius first law

$$X-iY=i\frac{\rho}{2}\oint_{C_0} w^2\,dz$$

$$w^2 = U^2 - \frac{2\mathrm{U}^2 a^2}{z^2} + \frac{U^2 a^4}{z^4} + \boxed{\frac{i U \Gamma}{\pi z}} - \frac{i U \Gamma a^2}{\pi z^3} - \frac{\Gamma^2}{4 \pi^2 z^2}$$

0 -2 -4 -1 -3 -2

Term order in z

$$a_{-1} = \frac{i U \Gamma}{\pi}$$

$z = 0$ – sole isolated singularity of w^2, thus

$$X - iY = 2\mathrm{i}\pi \sum_k a_{-1,k} = 2\mathrm{i}\pi \frac{i U \Gamma}{\pi} = -i \rho U \Gamma$$

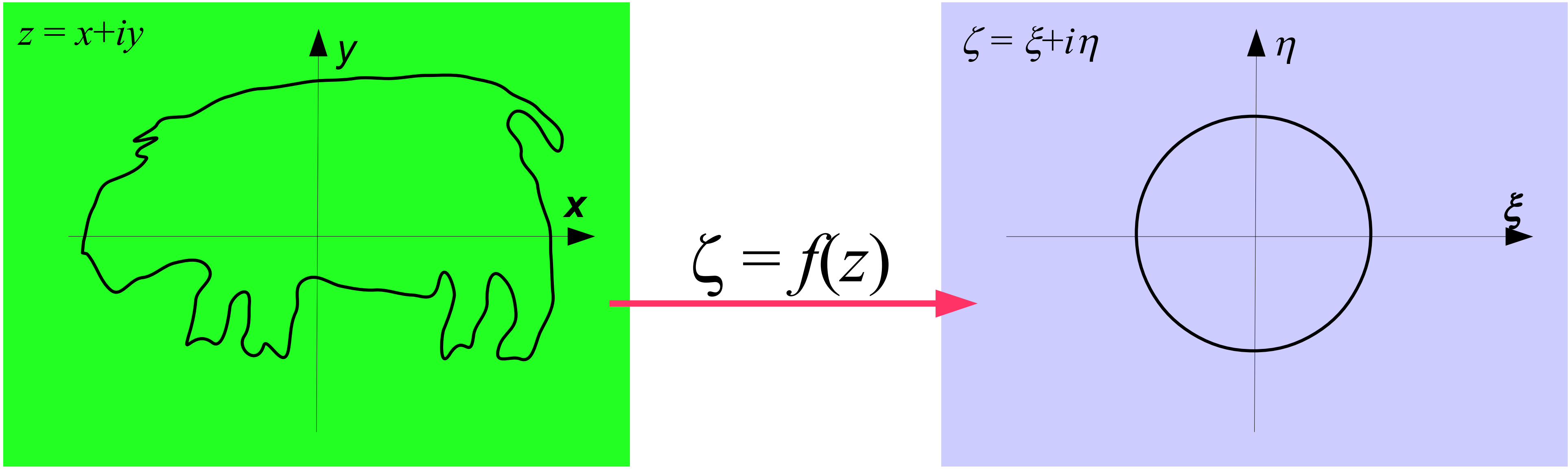
$X = 0$ (D'Alembert's paradox)

$Y = \rho U \Gamma$ (Zhukovsky-Kutta law)

Similar analysis for $z w^2$ produces $M = 0$

4.12. Conformal transformations

Helps deal with boundaries



It's only good if the Laplace equation is also transformed into something nice...

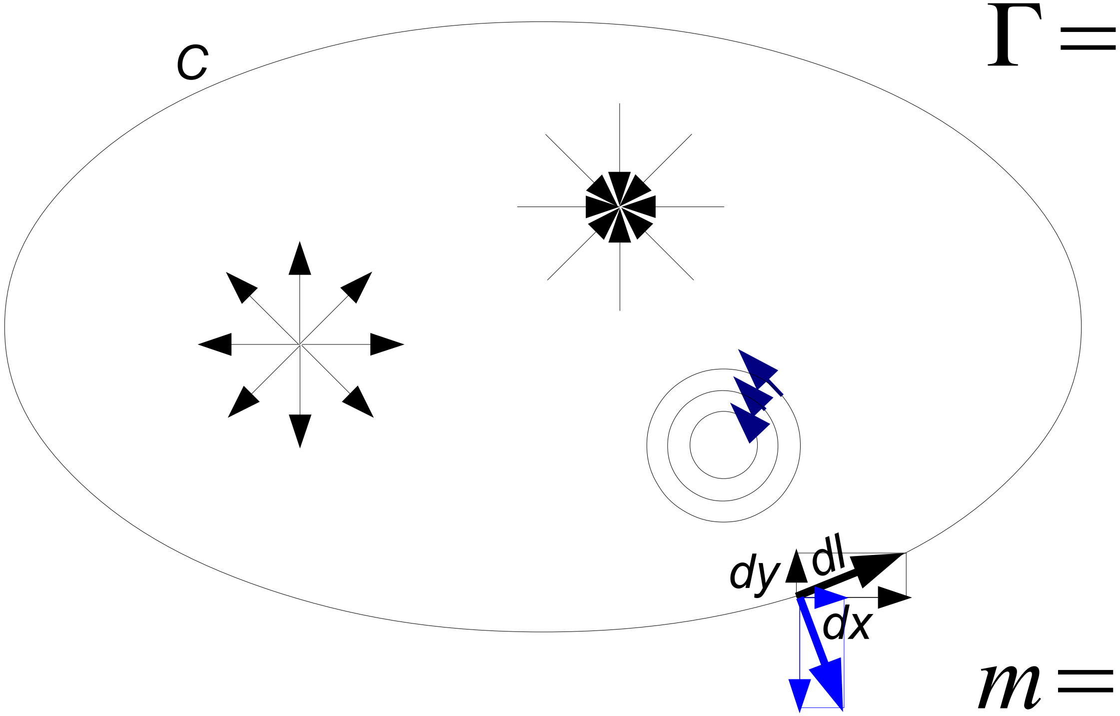
- Consider f – holomorphic function mapping (x,y) into (ξ,η)
- In (x,y) plane, let $\nabla^2\varphi(x,y) = 0$
- Then in (ξ,η) plane, $\nabla^2\varphi(\xi,\eta) = 0$

  (proof: p. 93)
- Laplace equation is preserved by **conformal mapping**
- What happens with complex velocity?

$$w(z)=\frac{dF}{dz}=\frac{dF(\zeta)}{d\zeta}\frac{d\zeta}{dz}=\frac{d\zeta}{dz}w(\zeta)$$

Velocity scales during conformal mapping

Let's prove that conformal mapping preserves sources, sinks, etc.



$$\Gamma = \oint_{C} \boldsymbol{u}\cdot d\boldsymbol{l} = \oint_{C} \left(u\,dx + v\,dy\right)$$

Circulation of all point vortices inside C

$$m = \oint_{C} \boldsymbol{u}\cdot d\boldsymbol{n} = \oint_{C} \left(u\,dy - v\,dx\right)$$

Strength of all sources/sinks inside C

$$\oint_C w(z)\,dz =$$

$$= \oint_C (u - iv)(dx + idy) = \oint_C (u\,dx + v\,dy) + i\oint_C (udy - vdx) =$$

$$= \Gamma + i\,m$$

Could have proven the same with residue theorem...

Now consider a conformal mapping $(x,y) \rightarrow (\xi,\eta)$

$$\left(\Gamma + i\,m\right)\Big|_z = \oint_{C|_z} w(z)\,dz =$$

$$= \oint_{C|_z} w(\zeta)\frac{d\,\zeta}{dz}dz =$$

$$= \oint_{C|_\zeta} w(\zeta)\,d\,\zeta = \left(\Gamma + i\,m\right)\Big|_\zeta$$

Conformal mapping preserves strength of sources, sinks, and vortices

4.13. Zhukovsky transformation

$$z=\zeta+\frac{c^2}{\zeta}$$

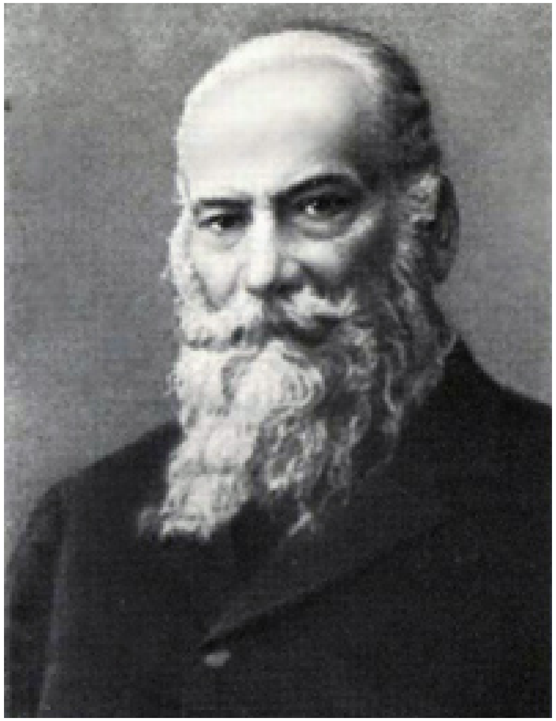
$$|\zeta|\rightarrow\infty,\quad z\rightarrow\zeta$$

$$\frac{dz}{d\zeta}=1-\frac{c^2}{\zeta^2}$$

$\zeta=0$: singularity (let's contain it inside the body)

$$\zeta=\pm c,\quad \frac{dz}{d\zeta}=0$$

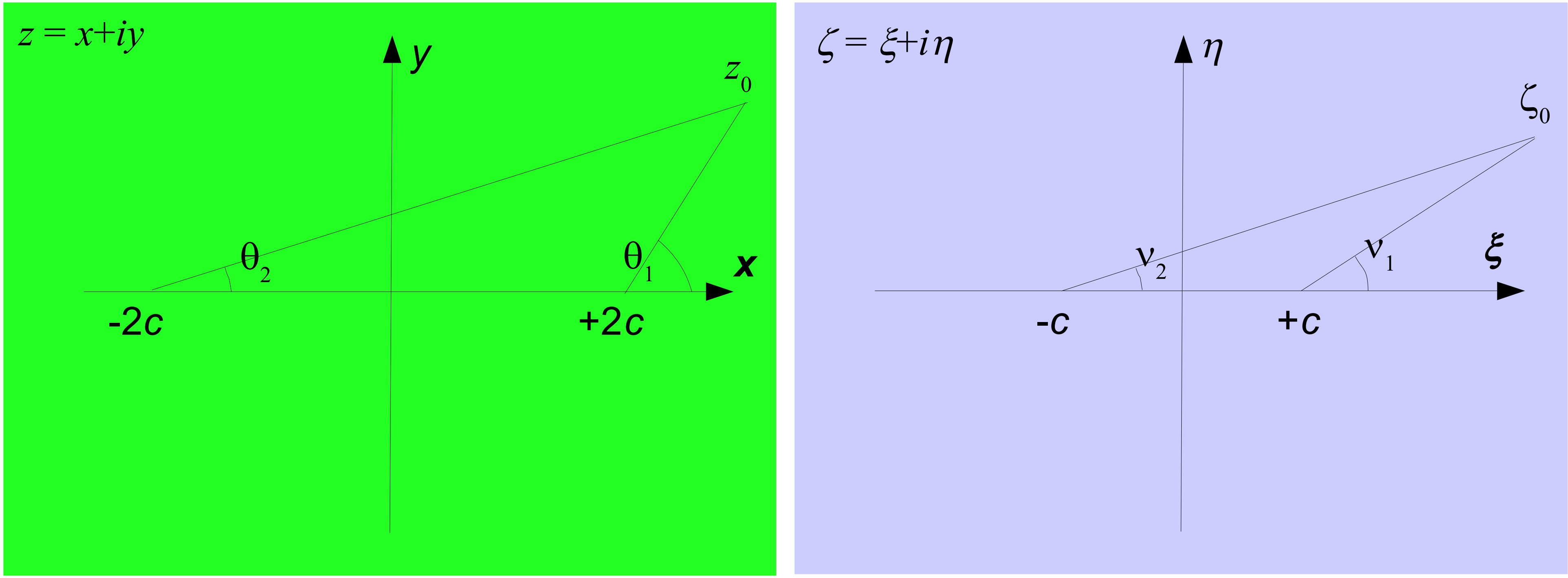
$\zeta=\pm c$: *critical points* (angle not preserved)



Nikolai Egorovich Zhukovsky (1847-1921)

“Man will fly using the power of his intellect rather than the strength of his arms.”

Critical points of Zhukovsky transform

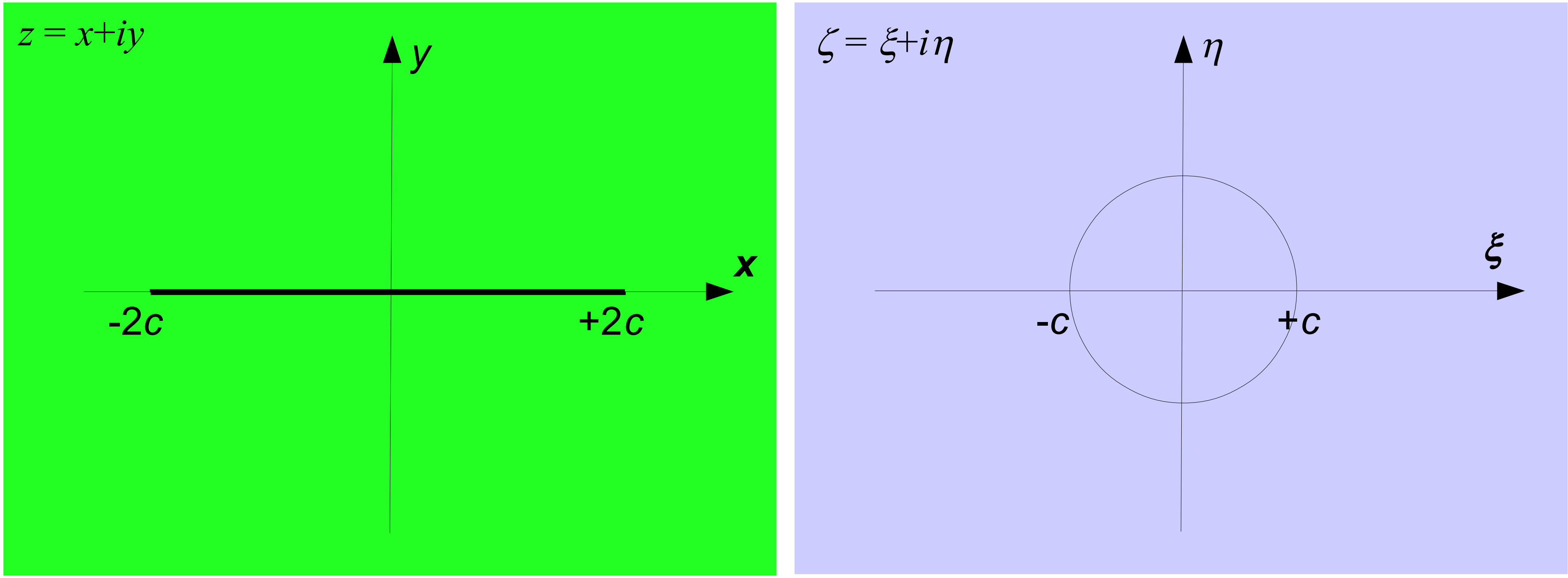


$$\zeta = \pm c \qquad z = \pm c + \frac{c^2}{\pm c} = \pm 2\mathrm{c}$$

Can prove: $\theta_1 - \theta_2 = 2\,(\nu_1 - \nu_2)$

A smooth curve passing through $\zeta = c$ will correspond to a curve with a cusp in z-plane

Example: $\zeta = ce^{i\nu}$



$$z = ce^{i\nu} + \frac{c^2}{ce^{i\nu}} = c\left(e^{i\nu} + e^{-i\nu}\right) = 2\mathrm{c}\cos\nu$$

Zhukovsky transform recipe. Start with flow around a cylinder in ζ-plane, map to something

4.14. Flow around ellipses

Circle in ζ-plane, radius $a > c$, center at origin

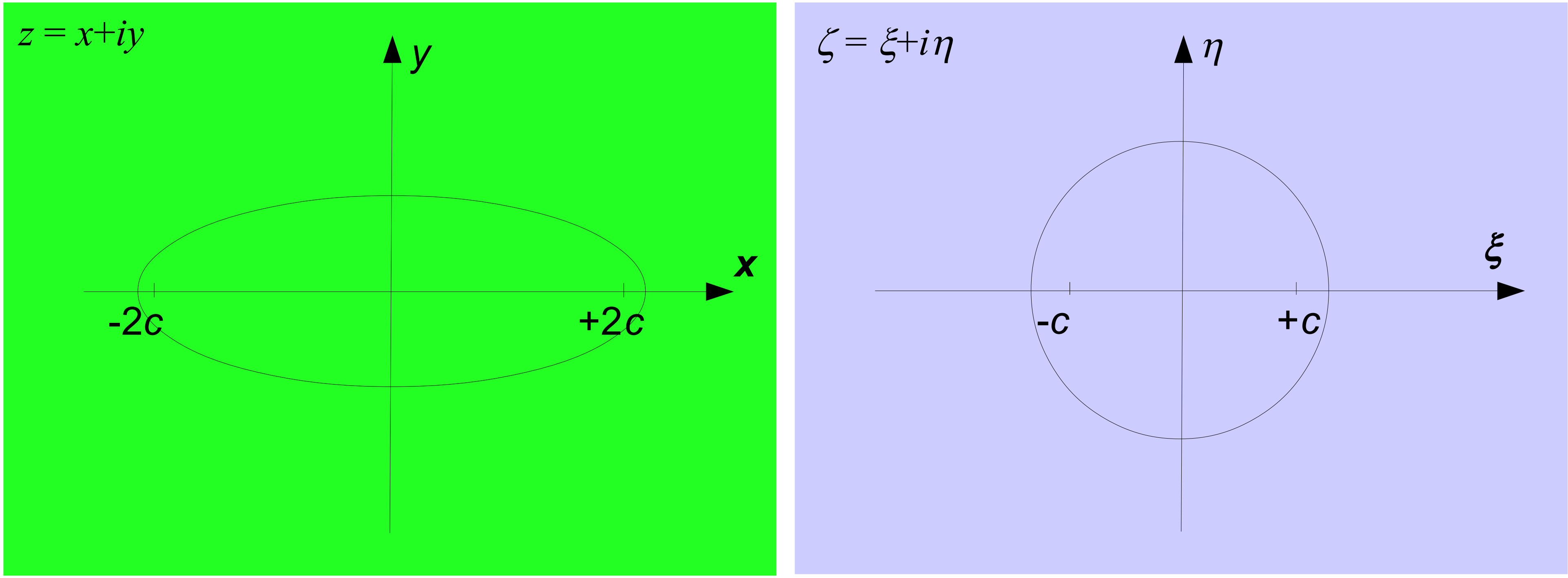
$$\zeta = a e^{i\nu}$$

$$z = a e^{i\nu} + \frac{c^2}{a} e^{-i\nu} = \left(a + \frac{c^2}{a}\right) \cos\nu + i \left(a - \frac{c^2}{a}\right) \sin\nu$$

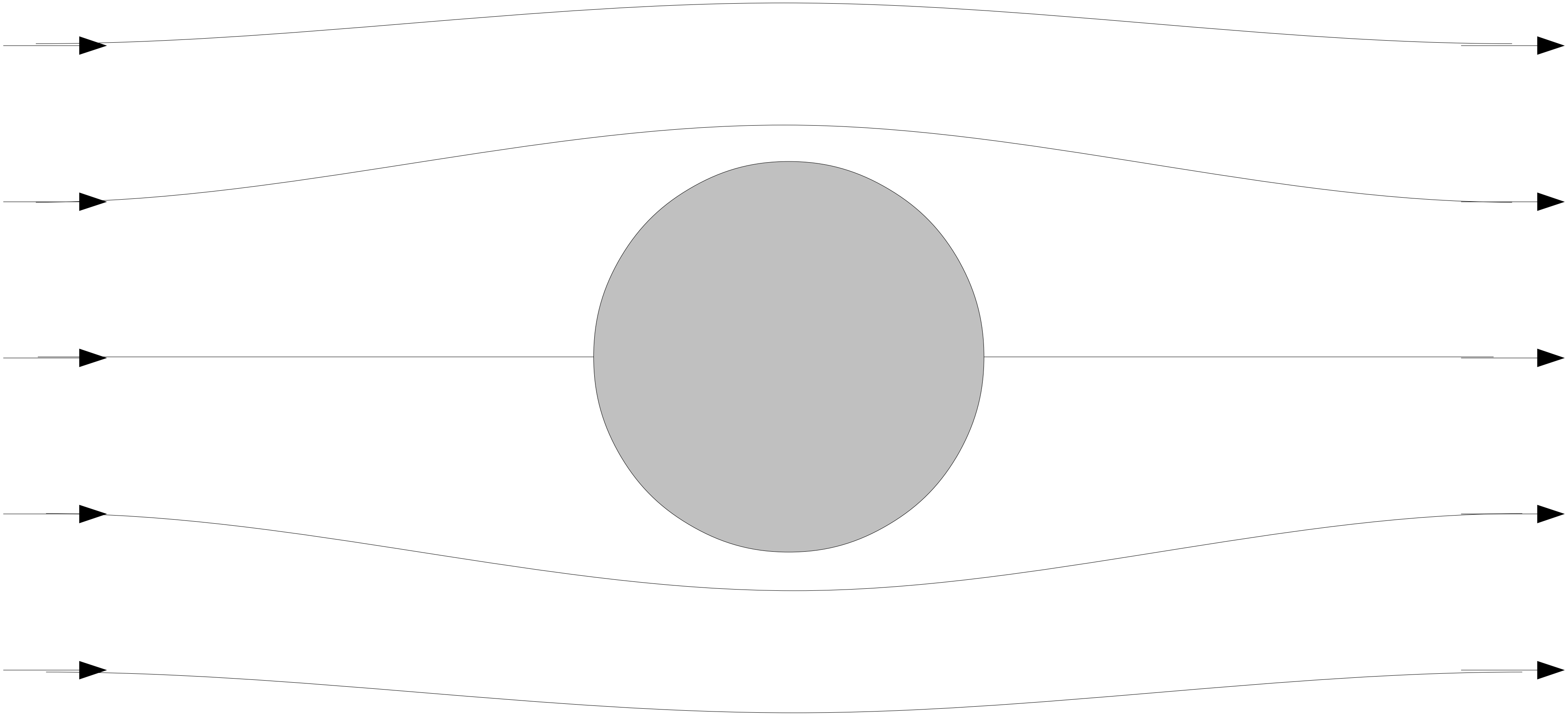
Major semiaxis

Minor semiaxis

Parametric equation of an ellipse


$z = x+iy$
y
x
$-2c$
$+2c$
$\zeta = \xi+i\eta$
η
ξ
$-c$
$+c$

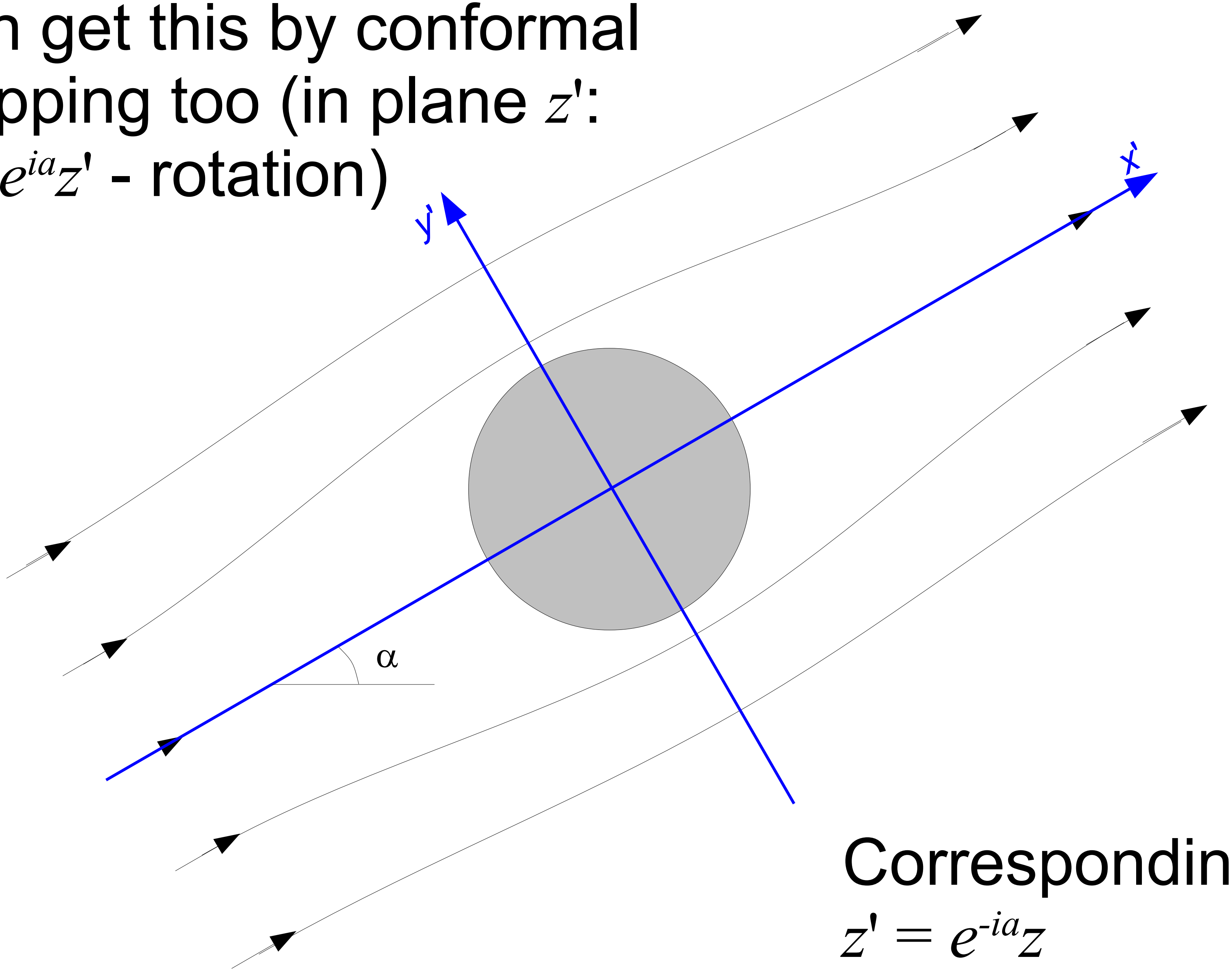
Flow past a cylinder



$$F(z) = U\left(z + \frac{a^2}{z}\right)$$

Now consider freestream flow at an angle

Can get this by conformal mapping too (in plane z': $z = e^{ia}z'$ - rotation)



Correspondingly, $z' = e^{-ia}z$

In plane z'

$$F(z')=U\left(z'+\frac{a^2}{z'}\right)$$

$$F=U\left(z\,e^{-i\alpha}+\frac{a^2}{z\,e^{-i\alpha}}\right)=U\left(z\,e^{-i\alpha}+\frac{a^2}{z}\,e^{i\alpha}\right)$$

Let's have this flow in ζ-plane:

$$F(\zeta)=U\left(\zeta\,e^{-i\alpha}+\frac{a^2}{\zeta}\,e^{i\alpha}\right)$$

Now recall that

$$z=\zeta+\frac{c^2}{\zeta}$$

Express ζ in terms of z:

$$\zeta^2 + c^2 - \zeta z = 0$$

$$\zeta = \frac{z}{2} \pm \sqrt{\left(\frac{z}{2}\right)^2 - c^2}$$

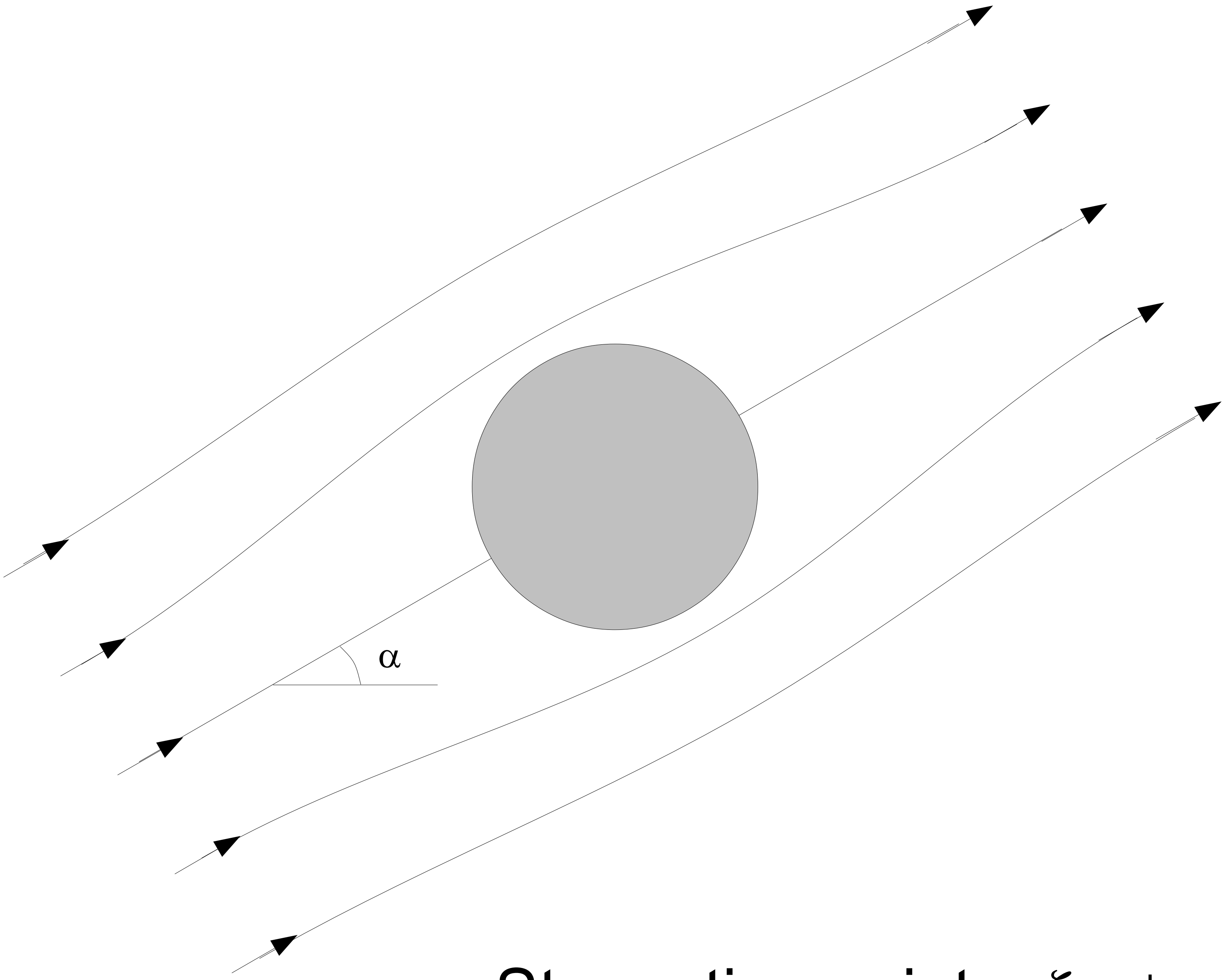
Recall that for $z \to \infty$, $z \to \zeta$. Thus select

$$\zeta = \frac{z}{2} + \sqrt{\left(\frac{z}{2}\right)^2 - c^2}$$

Plug this into $F(\zeta)$ to get $F(z)$... (skip derivation)

$$F(z)=U\left[ze^{-i\alpha}+\left(\frac{a^2}{c^2}e^{i\alpha}-e^{-i\alpha}\right)\left(\frac{z}{2}-\sqrt{\left(\frac{z}{2}\right)^2-c^2}\right)\right]$$

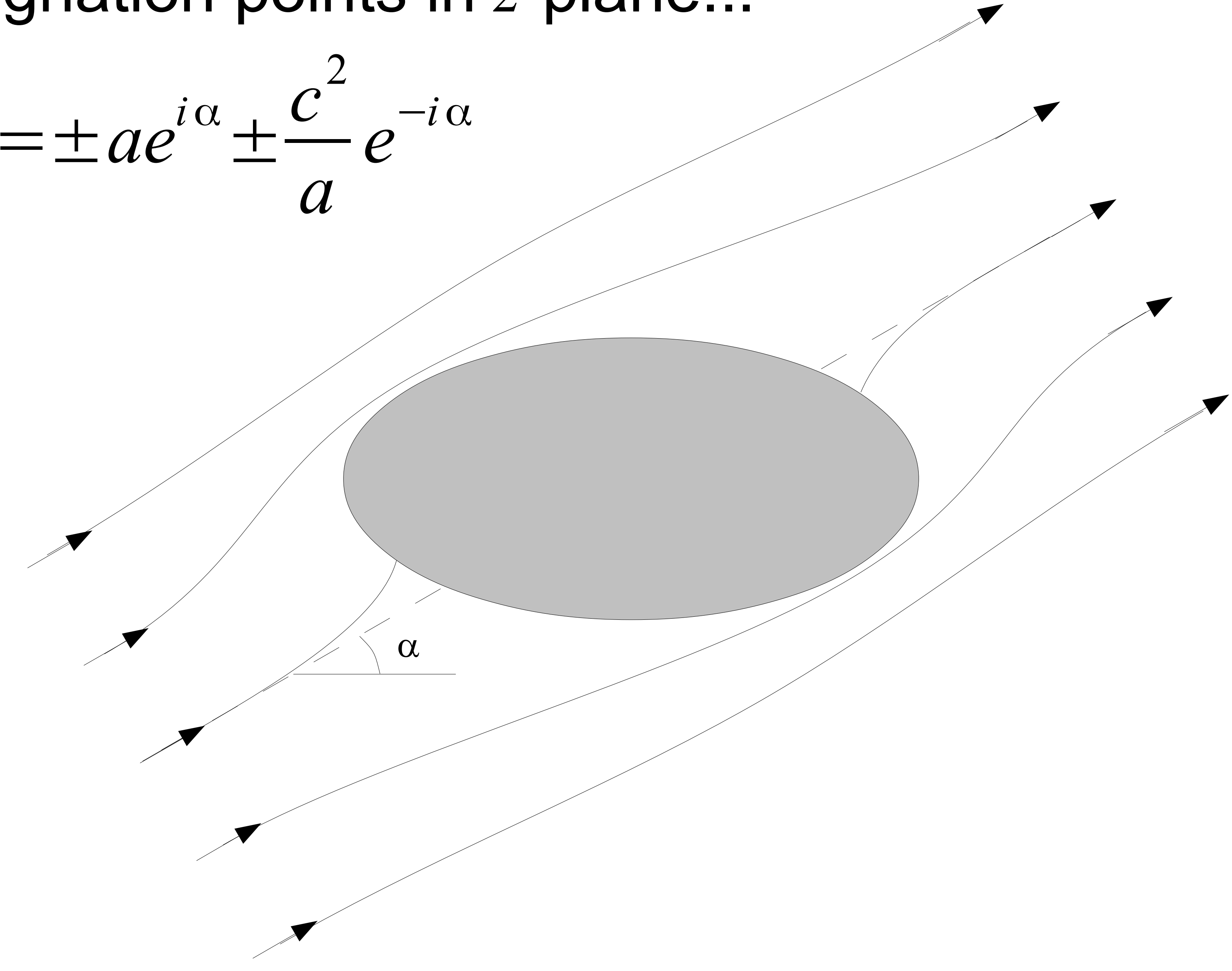
Uniform flow at angle α approaching an ellipse with major semiaxis $a + c^2/a$ and minor semiaxis a - c^2/a



Stagnation points: $\zeta = \pm a e^{i\alpha}$

Stagnation points in *z*-plane...

$$z = \pm a e^{i\alpha} \pm \frac{c^2}{a} e^{-i\alpha}$$



$$z = \pm\left(a + \frac{c^2}{a}\right)\cos\alpha \pm i\left(a - \frac{c^2}{a}\right)\sin\alpha$$

$$x = \pm\left(a + \frac{c^2}{a}\right)\cos\alpha$$

$$y = \pm\left(a - \frac{c^2}{a}\right)\sin\alpha$$

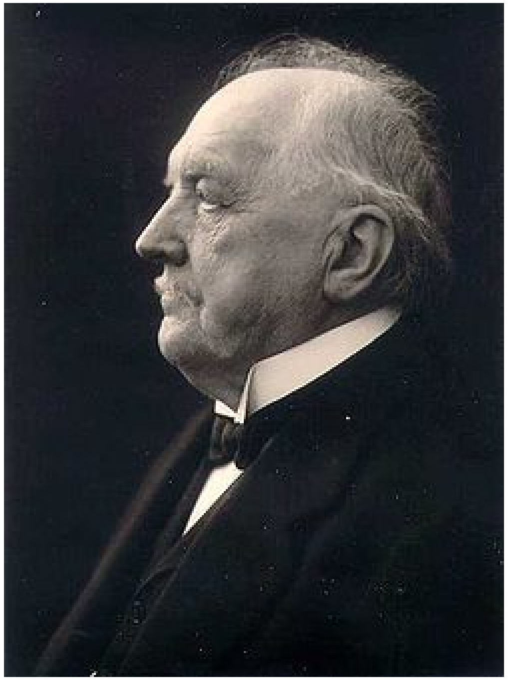
\- forward stagnation point

\+ downstream stagnation point

α = 0: horizontal flow approaching horizontal ellipse

$\alpha = \pi/2$: vertical flow, horizontal ellipse (or horizontal flow, vertical ellipse)

4.15. ~~Kutta condition~~ and the flat-plate airfoil



Martin Wilhelm Kutta
(1867-1944)

4.15. Zhukovsky-Chaplygin postulate and the flat-plate airfoil

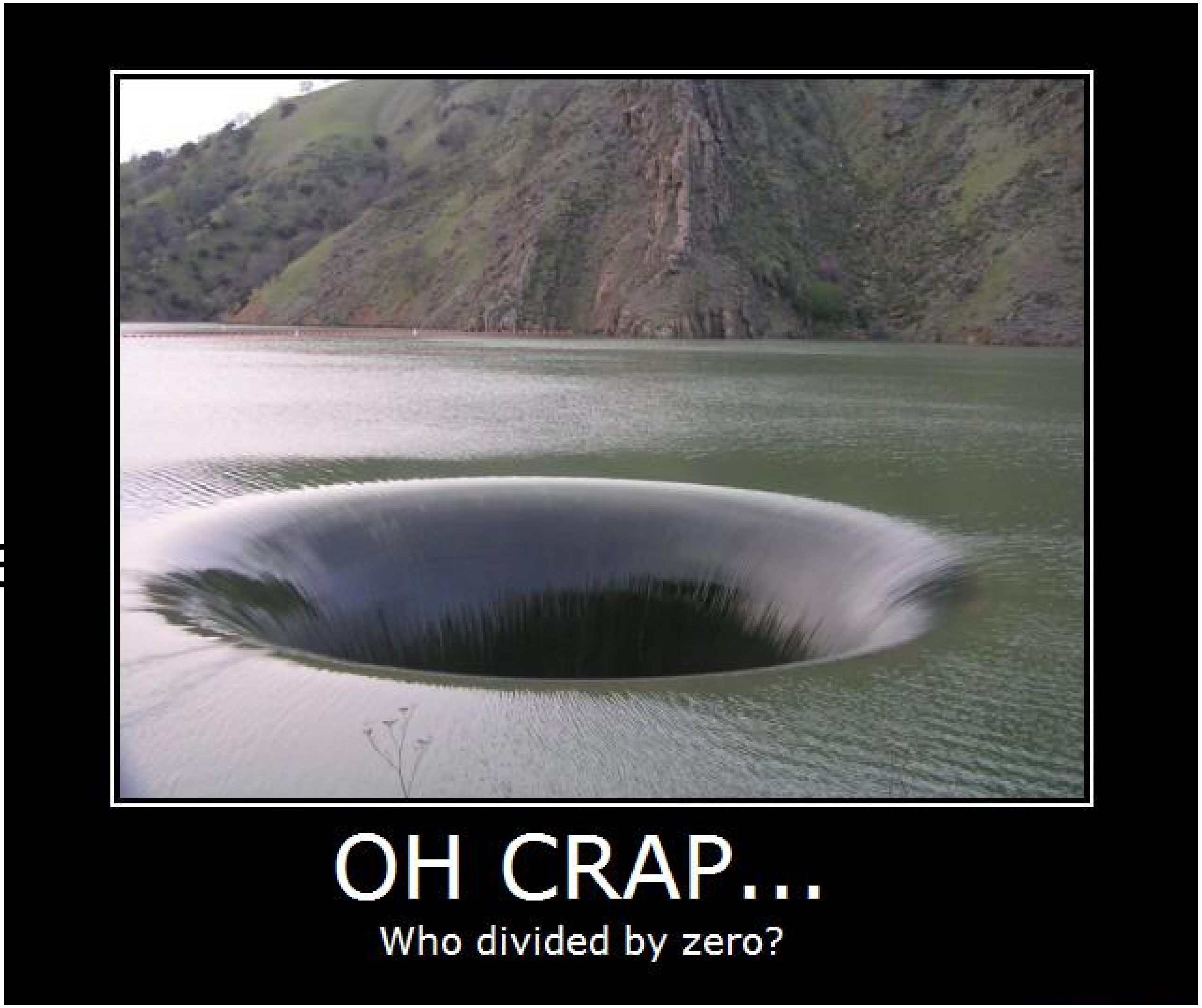
Sergey Chaplygin
(1869-1942),
Hero of Socialist Labour
(1 February 1941)

4.15. Zhukovsky-Chaplygin postulate and the flat-plate airfoil

Flow around a sharp edge (section 4.6)...

w

- At a sharp edge



4.15. Zhukovsky-Chaplygin postulate and the flat-plate airfoil

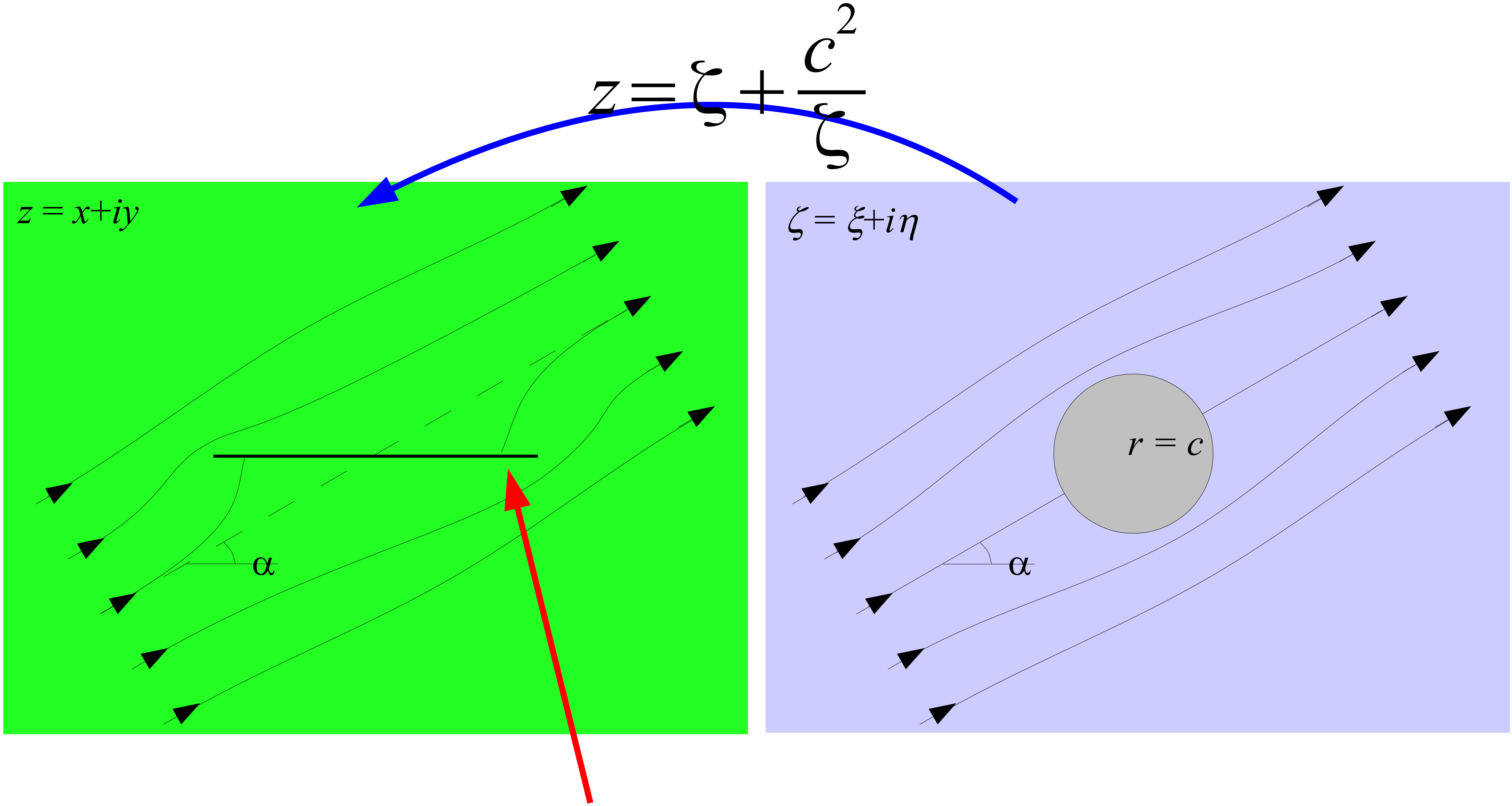
Flow around a sharp edge (section 4.6)...

$$F(z) = C\,z^{1/2}$$

$$w(z) = \frac{dF}{dz} = \frac{C}{2\,z^{1/2}}$$

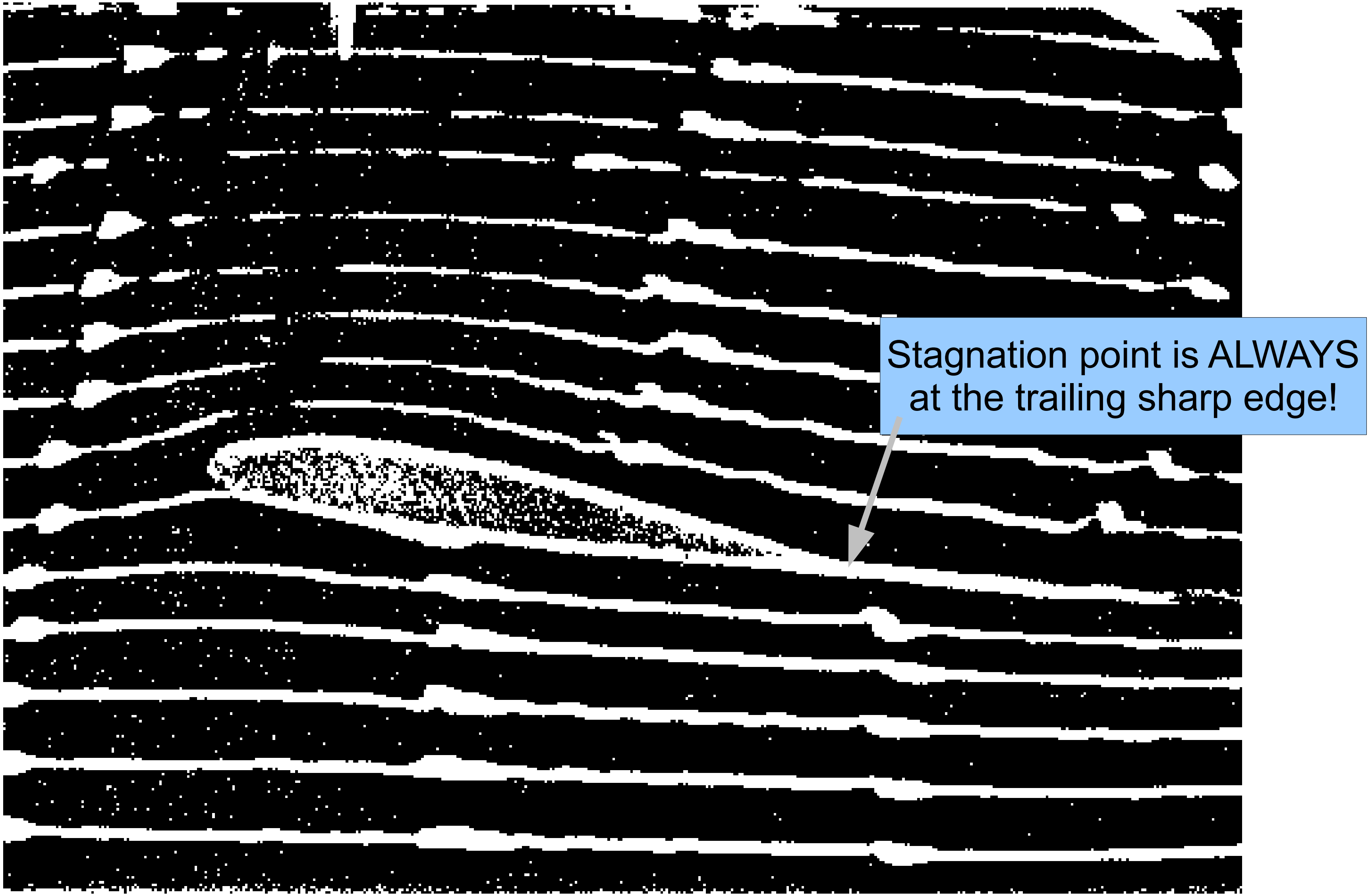
$z = 0$: singularity

- At a sharp edge, velocity goes to infinity
- This is not the case in experiment, luckily
- Need a fix for theory near sharp edges
- That's not the only problem though...



$$z = \zeta + \frac{c^2}{\zeta}$$

Herein lies
the problem!



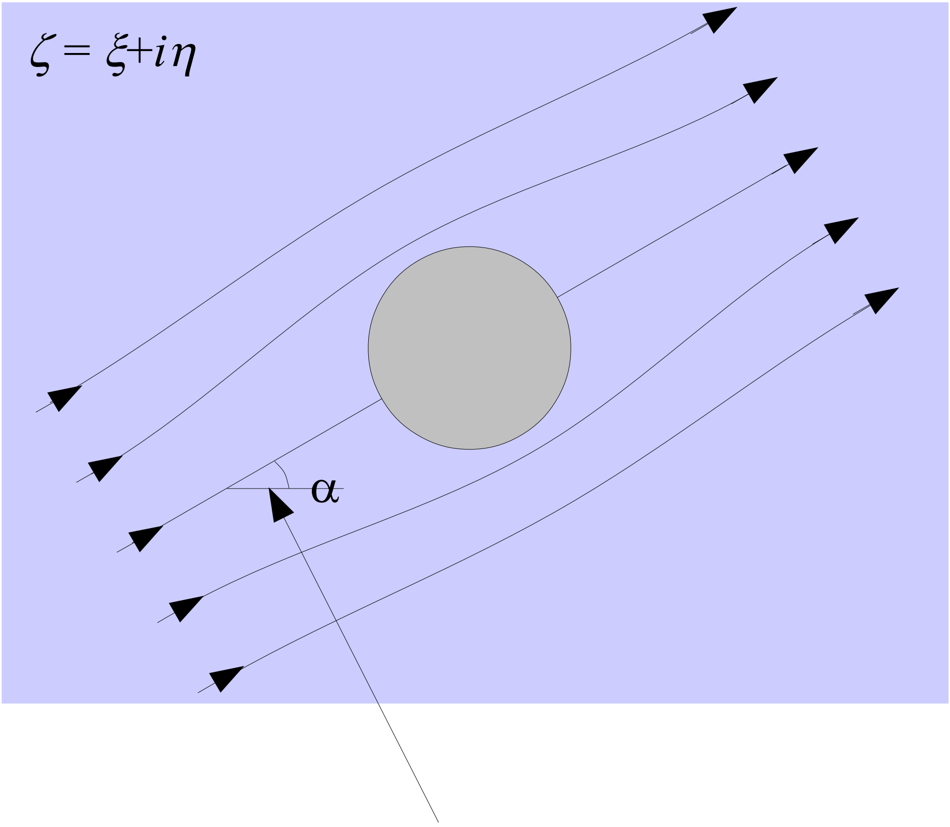
Stagnation point is ALWAYS at the trailing sharp edge!

Smoke visualization of wind tunnel flow past a lifting surface
Alexander Lippisch, 1953

Zhukovsky-Chaplygin postulate:
For bodies with sharp trailing edges at modest angles of attack to the freestream, the rear stagnation point will stay at the trailing edge

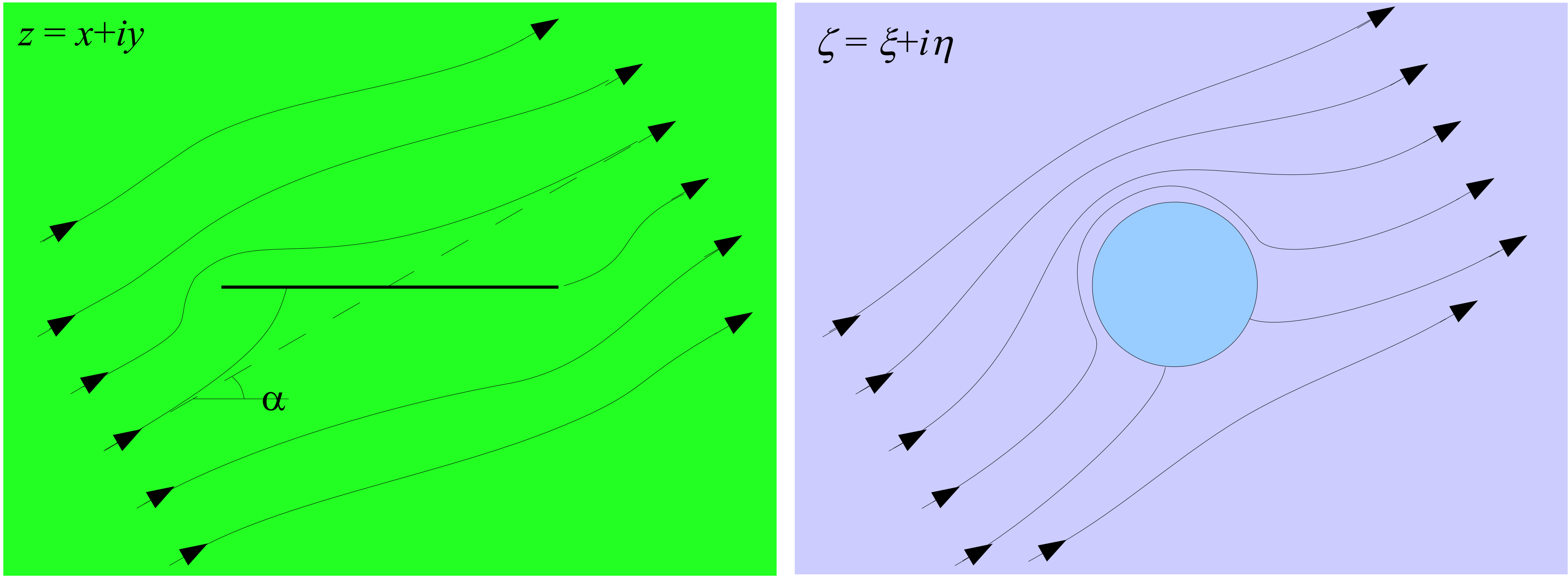
Dealing with trailing-edge singularity
In modeling real lifting surfaces, trailing edge has sharp but *finite* curvature

How to “fix” the flat-plate flow?



Angle of attack

Add circulation...



...to move the stagnation point to the trailing edge!

We want to move the rear stagnation point to $z = 2c$

That would correspond to $\zeta = c$ in the z-plane

Need to move it there from $\zeta = ce^{i\alpha}$

For cylinder flow with circulation...

$$\sin\theta_s = -\frac{\Gamma}{4\pi U a}$$

If $\sin\theta_s = -\sin\alpha$,

$$\Gamma = 4\pi U a \sin\alpha$$

Recipe for constructing a complex potential for corrected flat-plate flow (Eq. 4.22b)

- Cylinder flow
- Add circulation $\Gamma = 4\pi\, a\, U \sin \alpha$
- Rotate the plane α degrees counterclockwise
- Zhukovsky transform
- ???
- Profit!

Lift on a flat-plate airfoil extending from -$2a$ to $2a$

Blasius law for cylinder flow:

$$Y = \rho\, U\, \Gamma$$

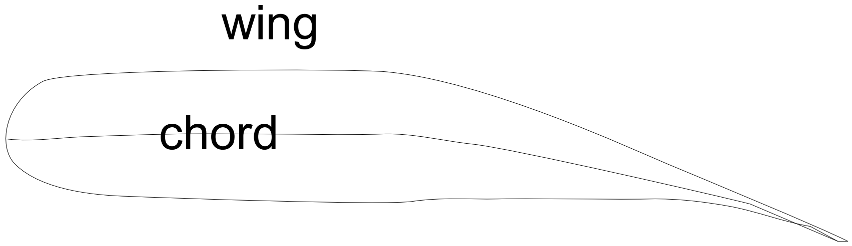
In our case

$$Y = 4\pi\rho U^2 a \sin\alpha$$

Introduce dimensionless *lift coefficient*

$$C_L = \frac{Y}{\frac{1}{2}\rho U^2 l}$$

Characteristic length scale
(for wings – *chord length*)



For our flat plate, $l = 4a$ and

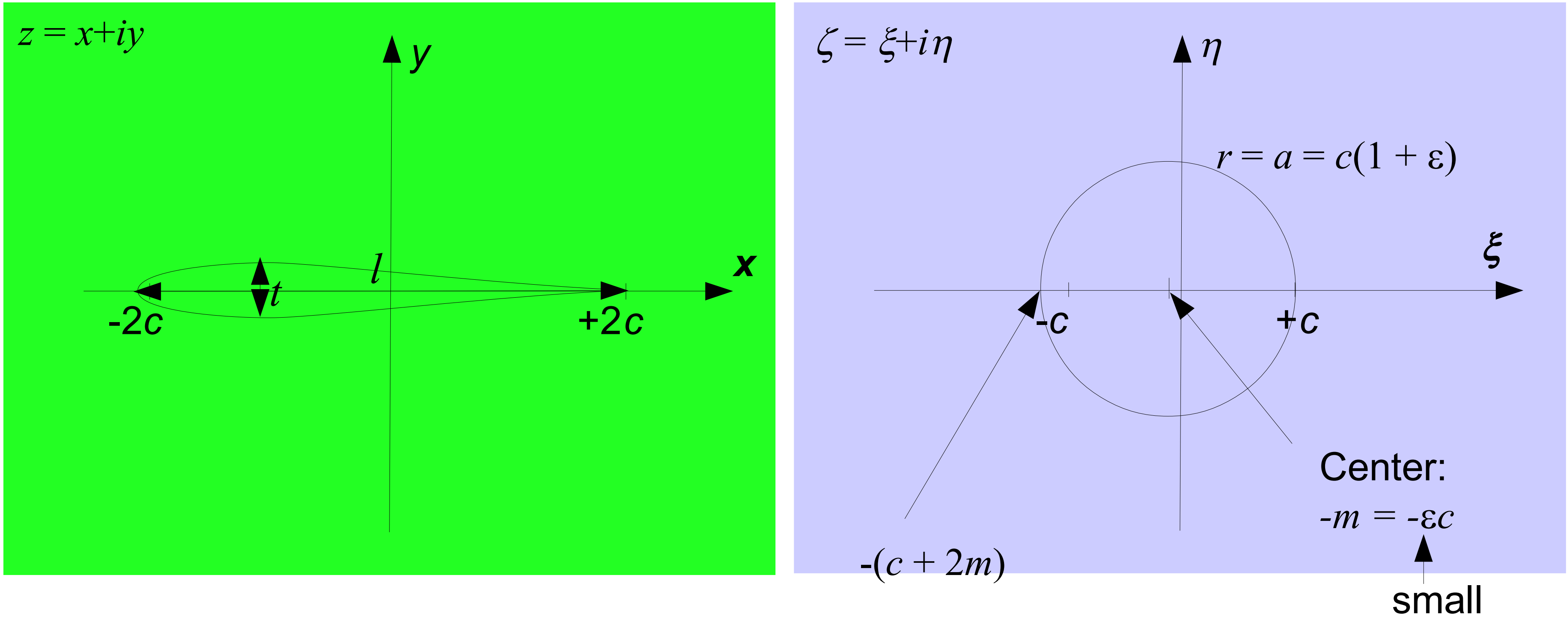
$$C_L = 2\pi \sin\alpha$$

At small angles of attack, lift coefficient on a flat plate increases with angle of attack!


САМОЛЕТ А.Ф. МОЖАЙС
A.F. MOZHAISKI AEROPL
1884 г.
1884
Модель. М 1:30
автор В.Г. Самойленко
Model. Scal
Author V.G.S

4.16. Symmetrical Zhukovsky airfoil

Goal: airfoil with sharp trailing edge and blunt leading edge



Leading edge in ζ-plane: -$(c + 2m)$

In z-plane, the leading edge is...

$$z = -c(1+2\varepsilon) - \frac{c}{1+2\varepsilon} = -2\mathrm{c} + O(\varepsilon^2) \approx -2\mathrm{c}$$

Chord length $l = 4c$

Similarly (more series expansions, linearization) thickness

$$t = 3\sqrt{3}c\varepsilon, \quad \frac{t}{l} = \frac{3\sqrt{3}}{4}\varepsilon$$

Thickness ratio

Maximum thickness occurs at $x = -c$

Extra Flugzeugbau EA300, 1987, Walter Extra design, Zhukovsky wing profile

Can find ε in ζ-plane from desired l and t in z-plane:

$$\varepsilon = \frac{4}{3\sqrt{3}}\frac{t}{l} \approx 0.77\frac{t}{l}$$

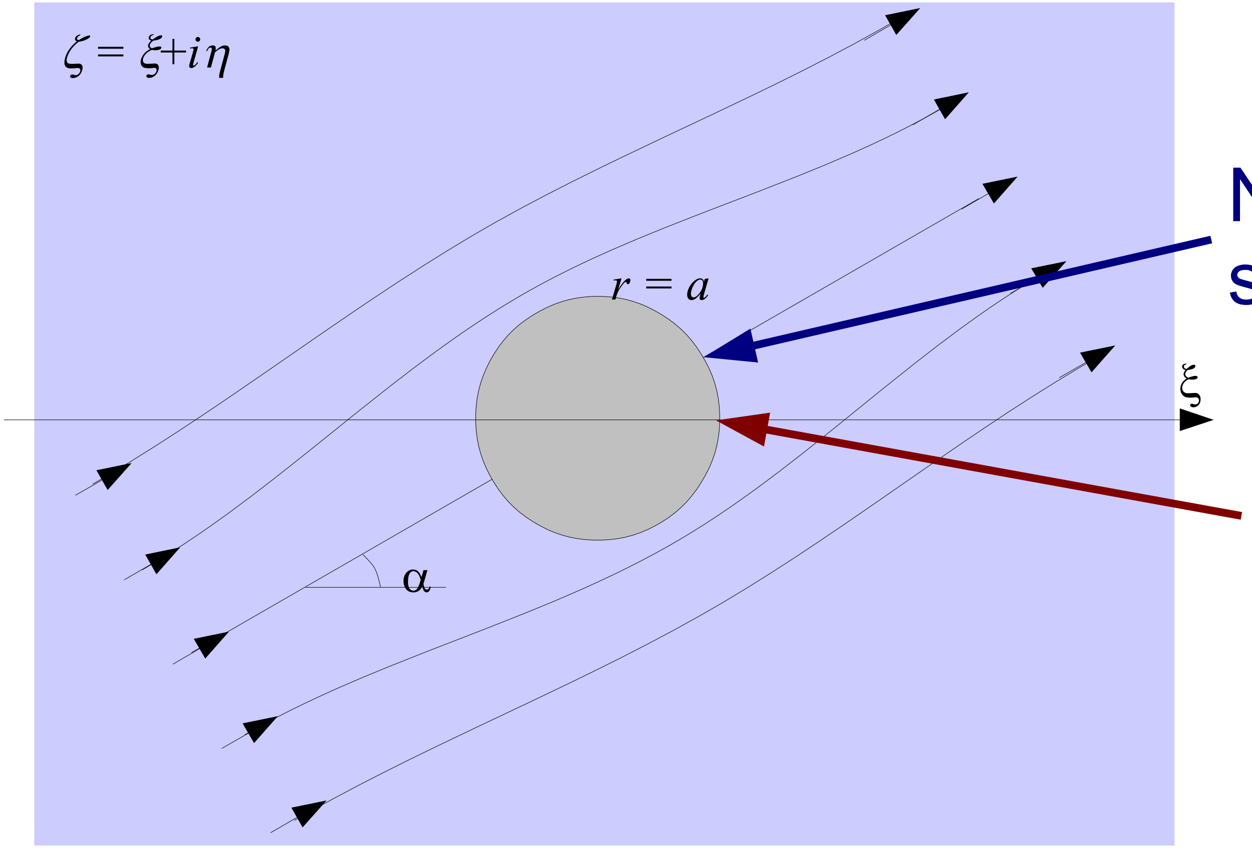
Equation for symmetric Zhukovsky profile in z-plane

$$\frac{y}{l} = \pm\frac{2}{3\sqrt{3}}\left(1-2\frac{x}{l}\right)\sqrt{1-\left(2\frac{x}{l}\right)^2}$$

At zero angle of attack, stagnation point is at trailing edge, lift = 0

Add angle of attack α...

To satisfy the Zhukovsky/Kutta/whatever condition...



Need to move this stagnation point...

...here!

For a cylinder of radius a, the needed amount of circulation is (same as for flat plate...)

$$\Gamma = 4\pi\, a\, U \sin\alpha$$

Express radius a in terms of l and t...

$$a = c + m = c(1+\varepsilon) = \frac{l}{4}\left(1 + \frac{4}{3\sqrt{3}}\frac{t}{l}\right)$$

For an angle of attack α, circulation we need to add is...

$$\Gamma = 4\pi U a \sin\alpha = \pi U l\left(1 + \frac{4}{3\sqrt{3}}\frac{t}{l}\right)\sin\alpha$$

Lift coefficient for symmetrical Zhukovsky airfoil

$$C_L \approx 2\pi\left(1 + 0.77\frac{t}{l}\right)\sin\alpha$$

$t \rightarrow 0$, this reduces to lift coefficient of flat plate

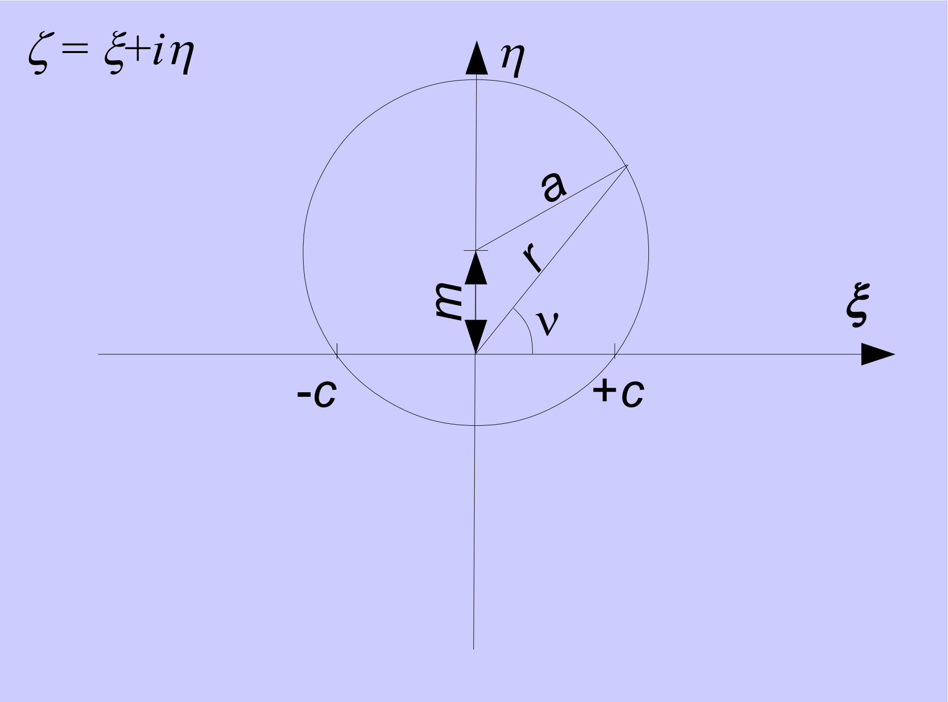
Zhukovsky symmetrical profile has better lift!

4.17. Arc airfoil

Airfoil of zero thickness but finite curvature

Use cosine theorem to get *r*

$$a^2 = r^2 + m^2 - 2\,r\,m\cos\left(\frac{\pi}{2} - \nu\right)$$



In z-plane,

$$z = r\,e^{i\nu} + \frac{c^2}{r}e^{-i\nu} =$$

$$= \overset{x}{\boxed{\left(r + \frac{c^2}{r}\right)\cos\nu}} + i\,\overset{y}{\boxed{\left(r - \frac{c^2}{r}\right)\sin\nu}}$$

$$x^2=\underset{\times\,\sin^2\nu}{\left(r^2+2\mathrm{c}^2+\frac{c^4}{r^2}\right)}\cos^2\nu,\quad y^2=\underset{\times\,\cos^2\nu}{\left(r^2-2\mathrm{c}^2+\frac{c^4}{r^2}\right)}\sin^2\nu$$

$$r^2\cos^2\nu\sin^2\nu=x^2\sin^2\nu-\left(2c^2+\frac{c^4}{r^2}\right)\cos^2\nu\sin^2\nu$$

$$\|$$

$$r^2\cos^2\nu\sin^2\nu=y^2\cos^2\nu+\left(2c^2-\frac{c^4}{r^2}\right)\cos^2\nu\sin^2\nu$$

$$x^2\sin^2\nu-y^2\cos^2\nu=4\mathrm{c}^2\cos^2\nu\sin^2\nu$$

Use cosine theorem:

$$\sin\nu=\frac{r^2-c^2}{2rm}=\left(r-\frac{c^2}{r}\right)\frac{1}{2m}=\frac{y}{2m\sin\nu}$$

y cannot be negative!!!

$$\sin^2 \nu = \frac{y}{2m}, \quad \cos^2 \nu = 1 - \frac{y}{2m}$$

$$x^2 \sin^2 \nu - y^2 \cos^2 \nu = 4\mathrm{c}^2 \cos^2 \nu \sin^2 \nu$$

$$x^2 \frac{y}{2m} - y^2 \left(1 - \frac{y}{2m}\right) = 4c^2 \frac{y}{2m} \left(1 - \frac{y}{2m}\right)$$

$$\frac{x^2}{2m} - y + \frac{y^2}{2m} = \frac{2c^2}{m} - c^2 \frac{y}{m^2}$$

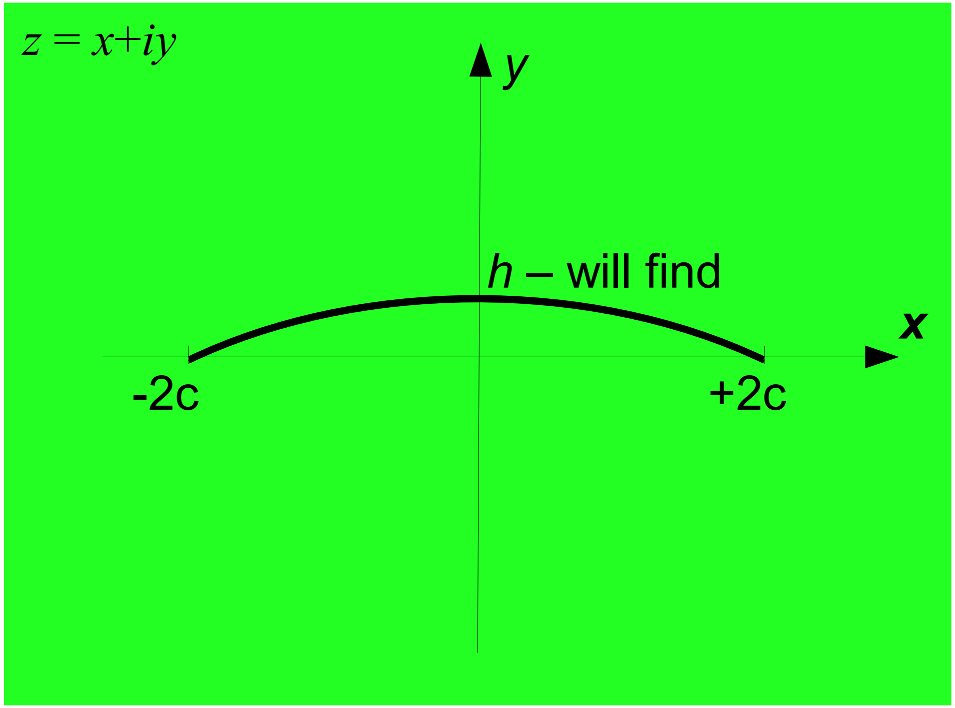
$$x^2 - 2my + y^2 = 4c^2 - 2c^2 \frac{y}{m}$$

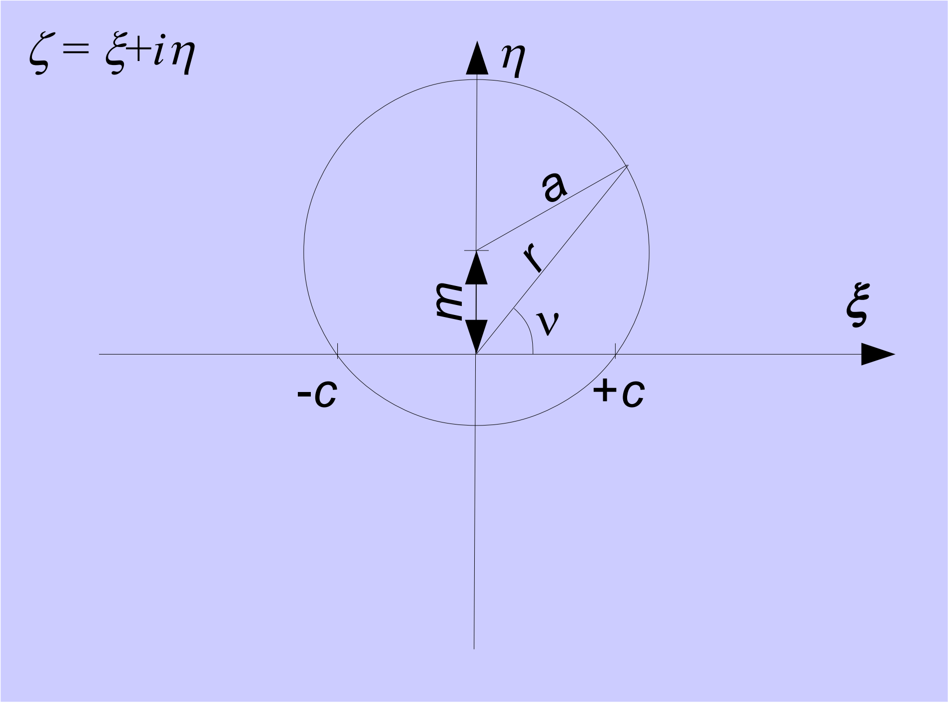
$$x^2 + \left[y + c\left(\frac{c}{m} - \frac{m}{c}\right)\right]^2 = c^2 \left[4 + \left(\frac{c}{m} - \frac{m}{c}\right)^2\right]$$

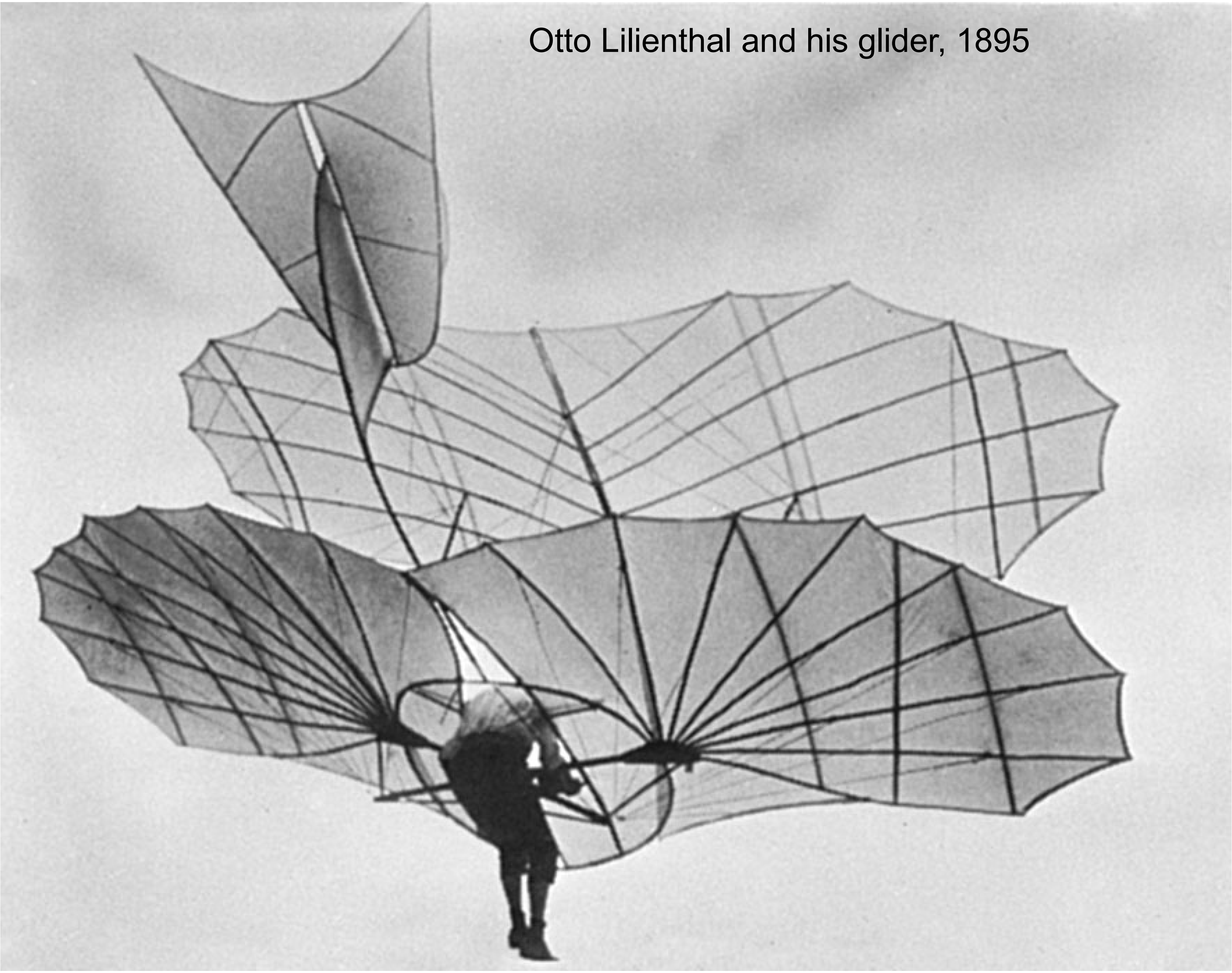
$$x^2+\left[y+c\left(\frac{c}{m}-\frac{m}{c}\right)\right]^2=c^2\left[4+\left(\frac{c}{m}-\frac{m}{c}\right)^2\right]$$

$$y\geqslant 0$$

Equation of an arc in the z-plane







Otto Lilienthal and his glider, 1895

$$x^2+\left[y+c\left(\frac{c}{m}-\frac{m}{c}\right)\right]^2=c^2\left[4+\left(\frac{c}{m}-\frac{m}{c}\right)^2\right]$$

Recall that $m/c = \varepsilon$, linearize (not essential here but nice)

$$x^2+\left(y+\frac{c^2}{m}\right)^2=c^2\left(4+\frac{c^2}{m^2}\right)$$

Find arc height h

Since $y = 2m\,\sin^2\nu$, $y_{\max} = h = 2m$

Next have to add circulation to put stagnation point at the trailing edge (trickier, because cylinder is moved upward in the ζ-plane)

Stagnation point needs to rotate by $\alpha + \tan^{-1}(m/c)$

Angle of attack Vertical shift

Linearize:

$$\tan^{-1}(m/c) \approx m/c = \varepsilon,\ a \approx c$$

Amount of circulation to be added:

$$\Gamma = 4\pi U a \sin\left(\alpha + \frac{m}{c}\right) \approx 4\pi U c \sin\left(\alpha + \frac{m}{c}\right)$$

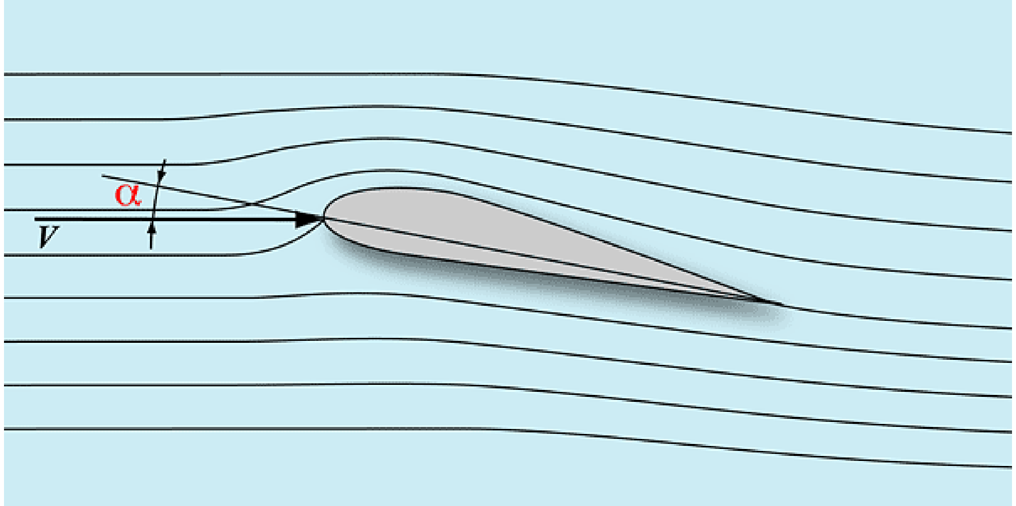
Lift coefficient:

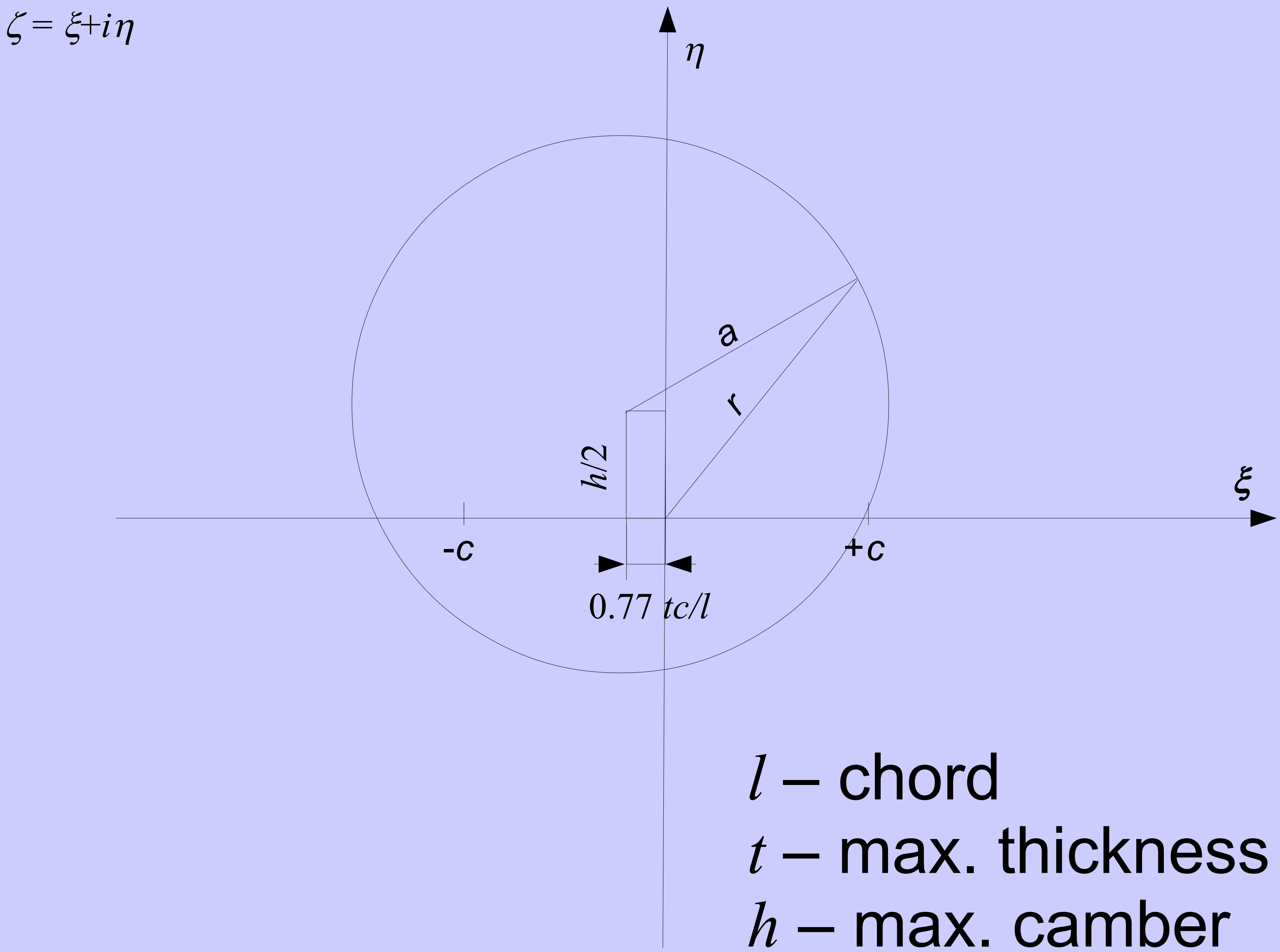
$$C_L = 2\pi U c \sin\left(\alpha + \frac{m}{c}\right) = 2\pi U c \sin\left(\alpha + 2\frac{h}{l}\right)$$

Again, more lift than flat plate!

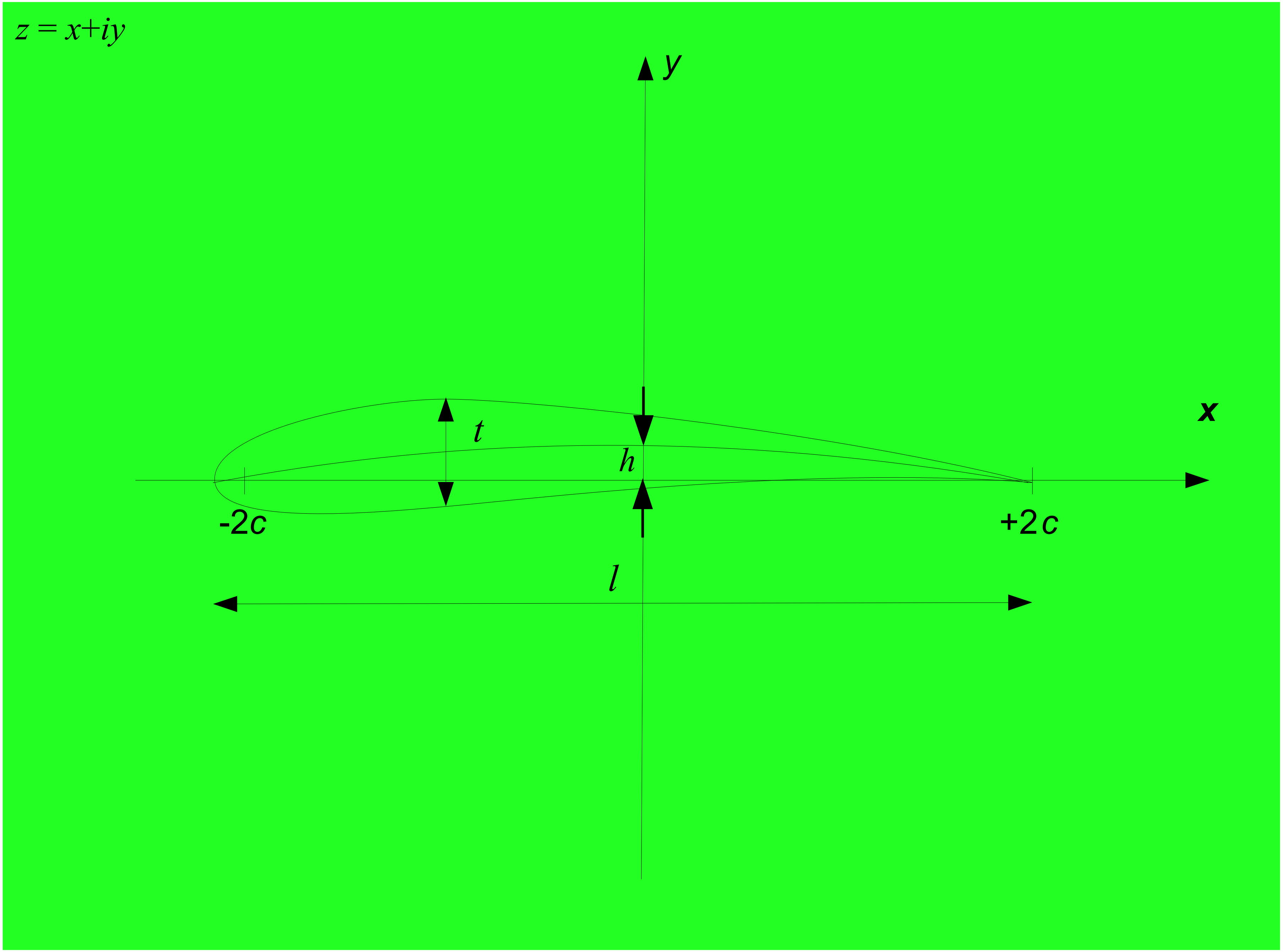
4. 18. Zhukovsky airfoil

- Know how to create lifting surfaces with:
  - Straight chord, finite thickness
  - Zero thickness, small finite curvature (camber)
- Both improve lift, compared with flat plate
- Create a lifting surface with **both** *thickness* and *camber* (Zhukovsky profile)

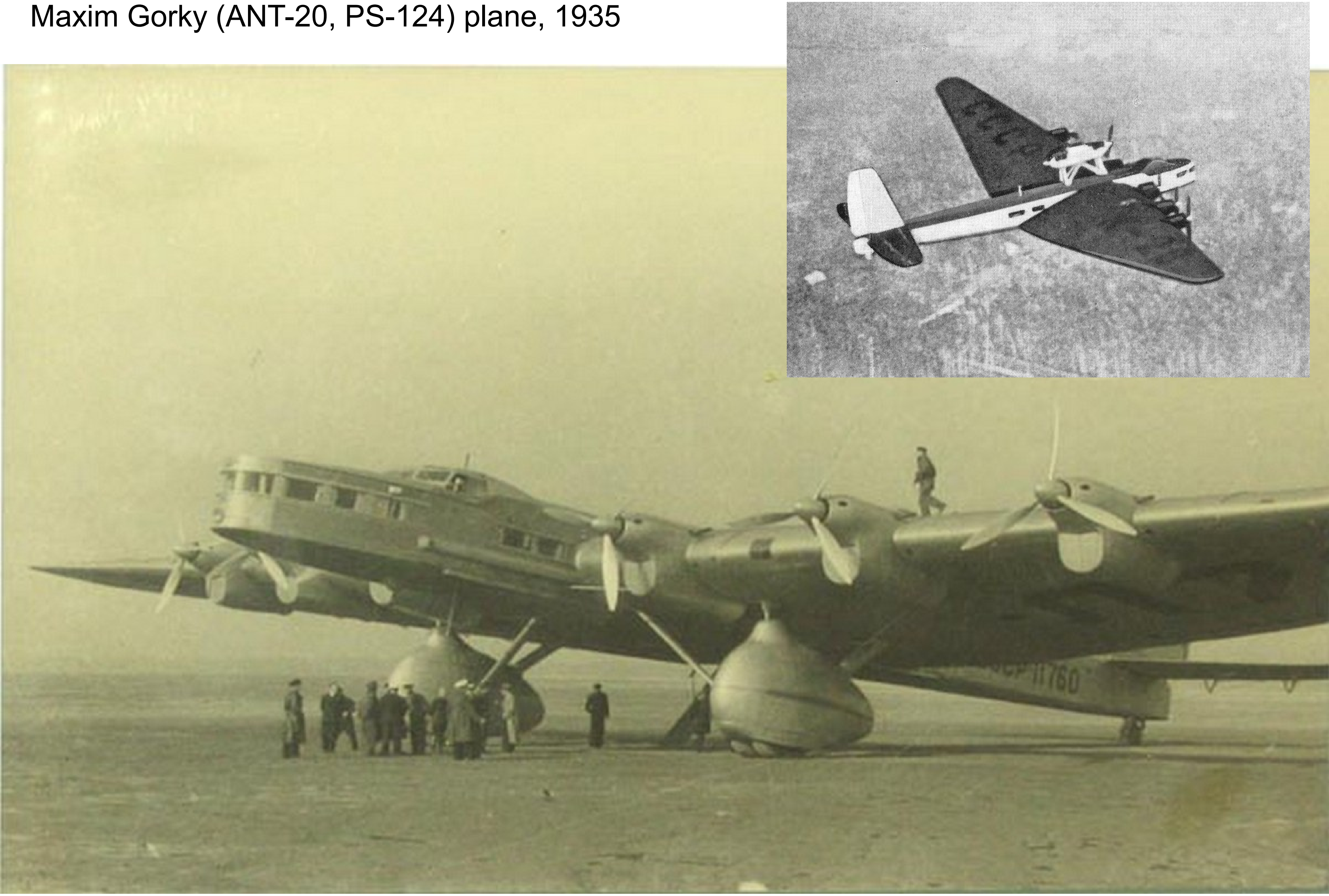




l – chord
t – max. thickness
h – max. camber


$z = x+iy$
y
x
t
h
$-2c$
$+2c$
l

Maxim Gorky (ANT-20, PS-124) plane, 1935



Circulation

$$\Gamma = \pi U l \left(1 + 0.77 \frac{t}{l} \right) \sin\left(\alpha + \frac{2h}{l} \right)$$

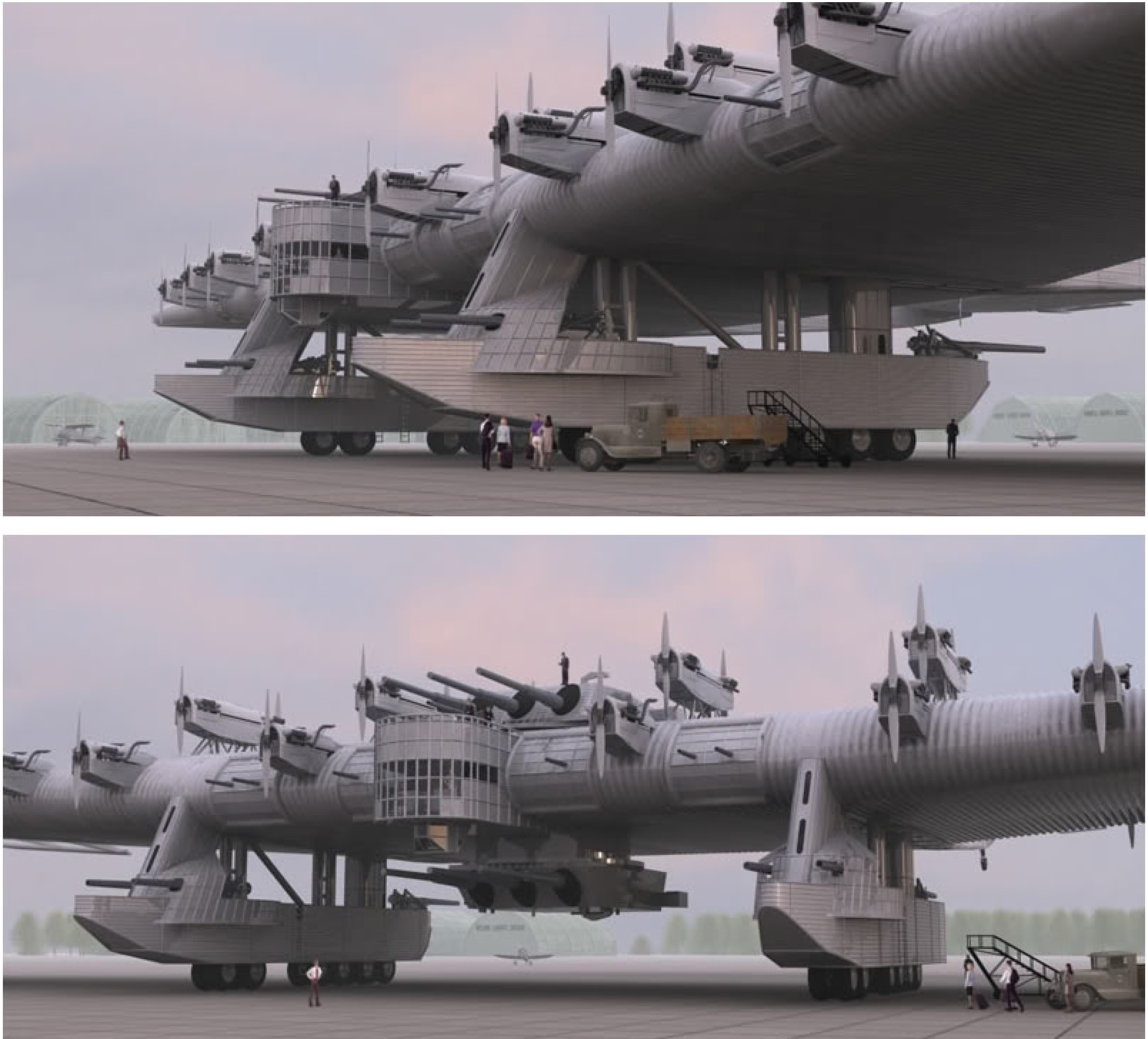
thick
ness

cam
ber

Lift coefficient

$$C_L = 2\pi \left(1 + 0.77 \frac{t}{l} \right) \sin\left(\alpha + \frac{2h}{l} \right)$$

Kalinin K-7 (Russia, 1930)



Dornier X flying boat (Germany, 1929)