Report 2 - Time-Dependent Advection-Diffusion

Add the advection-diffusion modules to the FE2D framework and complete the time-dependent advection-diffusion solver. Using the completed code, perform the computational studies outlined in parts (a), (b) and (c) below. Report your findings using the report format outlined for the first computational problem.

a) Advection of a Gaussian Distribution

Perform a verification study using the advection-diffusion solver in FE2D to solve

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \kappa\frac{\partial^2 T}{\partial x^2}, \tag{1}$$

on $-\infty < x < \infty$. The initial conditions are

$$T(x,0) = \exp\left\{-\frac{(x-\overline{x})^2}{2\sigma_0^2}\right\}, \tag{2}$$

where

$$\overline{x} = x_0 + \int_0^t u(\tau)d\tau, \tag{3}$$

and

$$\sigma^2 = \sigma_0^2 + 2\kappa t. \tag{4}$$

The exact solution is

$$\hat{T}(x,t) = \frac{\sigma}{\sigma_0}\exp\left\{-\frac{(x-\overline{x})^2}{2\sigma^2}\right\}. \tag{5}$$

Perform computations using two diffusivities: $\kappa = 0$ and $\kappa = 1$, with $T(0,t) = 0$, $x_0 = 50$, $\sigma = 5$, and $u = 1.0$ In order to verify that the advection-diffusion solver is functioning properly, use the exact solution for the problem. Check the numerical solution using the lumped, consistent and "high-order" mass matrices with a variety of time-integration options (i.e., $\theta = 0,\ 1/2,\ 1$) against the exact solution at $t = 50,\ 100,\ 200$ time units. Measure the error in your solution using both the $\mathcal{L}_1$ and $\mathcal{L}_2$ norms which, in discrete form, are

$$\mathcal{L}_1 = \frac{\sum_{i=1}^{Nnp} \mid T_i - \hat{T}(x_i) \mid}{\sum_{i=1}^{Nnp} \mid \hat{T}(x_i) \mid}, \tag{6}$$

and

$$\mathcal{L}_2 = \frac{\sum_{i=1}^{Nnp} [T_i - \hat{T}(x_i)]^2}{\sum_{i=1}^{Nnp} [\hat{T}(x_i)]^2} \tag{7}$$

From your numerical experiments, are there any difference between the time-integration methods or mass-matrix formulations? How sensitive is the solution to time step and mesh size? Summarize the results of the verification study in the report format outlined for the first computational problem. Be sure to report your grid resolution and the CFL number for each computation.

b) The Rotating Cone

Solve the 'pure' advection problem,

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = 0 \tag{8}$$

on $-100 \leq x \leq 100$, and $-100 \leq y \leq 100$, where $u = -\omega\ y$, $v = \omega\ x$, and $\omega = \pi/100$.

The exact solution to this problem is

$$\hat{T}(x, y, t) = \exp\left\{-\frac{(x-\overline{x})^2}{2\sigma_0^2} - \frac{(y-\overline{y})^2}{2\sigma_0^2}\right\}, \tag{9}$$

where

$$\overline{x} = x_0 + \int_0^t u(\tau)d\tau, \tag{10}$$

and

$$\overline{y} = y_0 + \int_0^t v(\tau)d\tau. \tag{11}$$

Perform computations using $\sigma = 10$ and $(x_0, y_0) = (50, 0)$. Check the numerical solution using the lumped, consistent and "high-order" mass matrices with a variety of time-integration options (i.e., $\theta = 0,\ 1/2,\ 1$) against the exact solution after $1, 2,$ and 5 rotations of the cone. Compute the $\mathcal{L}_1$ and $\mathcal{L}_2$ norms (defined above) at these times, along with the following two error measures.

Error Measure	Formula
Peak Error	$(T_{max} - \hat{T}_{max})/\hat{T}_{max}$
Maximum Negative Value	$\mid T_{max,neg}/\hat{T}_{max} \mid$

Discuss any pathologies that you may see in the computed solution.

c) Convection in Closed Streamlines

Solve

$$\frac{\partial T}{\partial t} + u\nabla T = \kappa \nabla^2 T \tag{12}$$

on $0 \leq x \leq 1$, and $0 \leq y \leq 1$, where the velocity is based on the stream function $\psi(x, y) = 0.08 \sin(4\pi x) \cos(4\pi y)$. The velocity components are

$$u = -\frac{\partial \psi}{\partial y} = 0.32\pi \sin(4\pi x) \sin(4\pi y), \tag{13}$$

and

$$v = \frac{\partial \psi}{\partial x} = 0.32\pi \cos(4\pi x) \cos(4\pi y). \tag{14}$$

The boundary conditions are

$$\kappa \frac{\partial T}{\partial x} = 0 \tag{15}$$

at $x = 0,\ 1$, along $0 \leq y \leq 1$ and

$$\kappa \frac{\partial T}{\partial y} = 0 \tag{16}$$

at $y = 0,\ 1$ along $0 \leq x \leq 1$.

The initial conditions are

$$T(x, y, t) = \exp\left\{ -\frac{(x - x_0)^2}{2\sigma_0^2} - \frac{(y - y_0)^2}{2\sigma_0^2} \right\}, \tag{17}$$

where $(x_0, y_0) = (0.5, 0.5)$, and $\sigma_0 = 0.1$. Use $\kappa = 10^{-3}$. Perform calculations using the lumped, consistent and "high-order" mass matrices with a variety of time-integration options (i.e., $\theta = 0,\ 1/2,\ 1$).