# Summary of the Blasting Technique for Imposing Essential Boundary Conditions

Consider the *Nnp* x *Nnp* linear equation

$$[K]\{T\}=\{Q\} \quad 1$$

having an essential boundary condition

$$T_k=\hat{T}\ . \quad 2$$

We can impose this boundary condition on the linear equation by making the following substitutions:

$$\begin{aligned} Q'_k &= \lambda k_{k,k}\hat{T} \\ k'_{k,k} &= \lambda k_{k,k} \end{aligned}\ . \quad 3$$

to form the approximation

$$[K']\{T\}\approx\{Q'\}\ . \quad 4$$

As justification, consider the dot product represented by the $k^{th}$ row of the approximation.

$$\sum_{k,j=1,n,j\neq k} k_{k,j}T_j+\lambda k_{k,k}\hat{T}=Q_k \quad 5$$

If $\lambda$ is sufficiently large, then the $\lambda k_{k,k}\hat{T}$ term overwhelms, or blasts, the summation so that

$$Q_k\approx Q'_k=\lambda k_{k,k}\hat{T}\ . \quad 6$$

It is possible to replace the $k^{th}$ term in the $Q$ vector with $\lambda k_{k,k}\hat{T}$. The resulting linear relationship now incorporates approximations of the boundary condition on both sides of the linear equation.

The question now is the quality of the approximation. Consider the residual factor, *r*, in equation 7. If $\lambda$ is sufficiently large, then *r* becomes small.

$$1-\frac{\lambda k_{k,k}\hat{T}}{Q_k}=r \quad 7$$

It can be shown that *r* provides a good measure of accuracy for the estimate. It has the advantage that it involves only terms known beforehand.

While blasting suffices to insert the boundary condition, it results in a matrix with a very large condition number, making equation 4 difficult to solve accurately. The solution is to normalize the equation. Perform the following manipulations

$$[W]^T [K'][W]^{-1} [W]\{T\} = [W]^T \{Q'\} \qquad 8$$

where $[W]$ is a specific diagonal matrix.

$$[W] = \begin{bmatrix} 1/\sqrt{k_{1,1}} & 0 & 0 & \dots & 0 \\ 0 & 1/\sqrt{k_{2,2}} & 0 & \cdots & 0 \\ 0 & 0 & 1/\sqrt{\lambda k_{k,k}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1/\sqrt{k_{n,n}} \end{bmatrix} \qquad 9$$

Recall that for a diagonal matrix,

$$[W] = [W]^T \qquad 10$$

Make the following substitutions

$$\begin{aligned} \left[\tilde{K}\right] &= [W]^T [K'][W] \\ \left\{\tilde{T}\right\} &= [W]^{-1} \{T\} \\ \left\{\tilde{Q}\right\} &= [W]\{Q\} \end{aligned} \qquad 11$$

so that we have a new linear equation:

$$\left[\tilde{K}\right]\left\{\tilde{T}\right\} = \left\{\tilde{Q}\right\} \qquad 12$$

To see what happens, look at how $\left[\tilde{K}\right]$ is formed. Consider the fixed location $k,l$ of $\left[\tilde{K}\right]$. The dot products of that form the $\tilde{k}_{k,l}$ term are

$$\tilde{k}_{k,l} = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} w_{k,j} k'_{j,l} \right) w_{i,l} \qquad 13$$

For the interior sum, $w_{k,j}$ will equal zero everywhere except when $k=j$. Similarly, $w_{i,l}$ will equal zero everywhere except when $i = l$.

$$\tilde{k}_{k,l} = \frac{1}{\sqrt{k'_{k,k}}} k'_{k,l} \frac{1}{\sqrt{k'_{l,l}}} = \frac{k'_{k,l}}{\sqrt{k'_{k,k} k'_{l,l}}} \qquad 14$$

On the diagonals, $k=l$ and

$$\tilde{k}_{k,k} = \frac{k'_{k,k}}{\sqrt{k'_{k,k} k'_{k,k}}} = 1 \qquad 15$$

Off the diagonal, if the row incorporates a boundary condition then

$$\tilde{k}_{k,l} = \frac{k'_{k,l}}{\sqrt{\lambda k'_{k,k} k'_{l,l}}} \approx 0 \qquad 16$$

Otherwise,

$$\tilde{k}_{k,l} = \frac{k_{k,l}}{\sqrt{k_{k,k} k_{l,l}}} \qquad 17$$

The result of equation 17 depends on the values of the parameters on the right side. However, in general such normalization reduces the variation of the matrix population and improves the condition number. This makes it easier to solve for $\{\tilde{T}\}$.

Transform given $\{\tilde{T}\}$ to $\{T\}$

$$[W]\{\tilde{T}\} = [W][W]^{-1}\{T\} = \{T\} \qquad 18$$