Report 3 - Transient Heat Conduction

Extend the one-dimensional finite element code, "heat1d" to solve the transient heat conduction equation,

$$\rho C \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left\{ k \frac{\partial T}{\partial x} \right\}, \tag{1}$$

where ρ is the mass density, C is the specific heat, and k is the thermal conductivity. In the formulation and implementation, use the generalized mass matrix,

$$\hat{M} = \alpha M^c + (1-\alpha) M^l, \tag{2}$$

where M^c is the consistent mass matrix, and M^l is the row-sum lumped mass matrix.

For the time integration algorithm, use the generalized time integration method which permits $0 \leq \theta \leq 1$. Include the capability to write the discrete solution, T^h at intermediate time intervals for plotting purposes.

Using the transient heat-conduction solver, consider a problem where an insulated steel rod of constant cross-section with a length of $1/4\ m$ is initially held in a tub of boiling water at 100^oC until the rod is at a uniform temperature. After achieving a uniform temperature, the rod is immersed in an ice bath at 0^oC.

a) Perform one verification computation using a grid with 10 linear elements. As a basis for comparison, use the exact solution,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \exp\left[-\frac{k}{\rho C}\left(\frac{n\pi}{L}\right)^2 t\right], \tag{3}$$

where

$$B_n = \frac{200}{n\pi}(1-\cos n\pi). \tag{4}$$

Compare your results at

$$\frac{k}{\rho C}\left(\frac{\pi}{L}\right)^2 t = \begin{cases} 1/4 \\ 1/2 \\ 3/4 \\ 1 \end{cases} \tag{5}$$

b) Using a mesh size $h = 0.0125$ (i.e., 20 elements) perform a series of computations varying the time integration weight θ and the mass weight α according to the table below.

Mass Matrix / Time Weight		
Consistent Mass	Lumped Mass	Higher-Order Mass
$\alpha = 1$ / $\theta = 0$	$\alpha = 0$ / $\theta = 0$	$\alpha = 1/2$ / $\theta = 0$
$\alpha = 1$ / $\theta = 1/2$	$\alpha = 0$ / $\theta = 1/2$	$\alpha = 1/2$ / $\theta = 1/2$
$\alpha = 1$ / $\theta = 1$	$\alpha = 0$ / $\theta = 1$	$\alpha = 1/2$ / $\theta = 1$

For each case in the table, plot the temperature profile as a function of position x at the times in Eq. (5) and discuss your results. As a basis for comparison, include a plot of exact solution at each time. Estimate the error between the exact and approximate solutions.

For all computations, use the following properties.

$\rho =$ Density ($7833\ kg/m^3$),

$C_p =$ Heat capacity ($465\ N \cdot m/kg \cdot^o C$),

$k =$ Thermal conductivity ($51.2564\ N \cdot m/s \cdot m \cdot^o C$),

$L =$ Rod length ($1/4\ m$).