Statics – Truss Problem

I. Statics

We are going to start our discussion with something very familiar. We are going to look at a simple statically determinate truss. The type of structure you analyzed in Statics. Trusses are characterized by linear elements (beams) that are pinned together at their ends. The pinned connection offers no resistance to rotation so there is no moment around the joint.

Consider the frame shown below:

The members in the structure are numbered with circles around them. We call these members elements. The points where the elements join are called nodes. In the problem above there are 5 elements and 4 nodes.

We can draw free body diagrams at the nodes.
In the free body diagrams above, we are using the variables $F$ for the forces in the elements. The subscripts on these forces refer to the element number. The variables $R$ represent the reactions. Their subscripts are assigned arbitrarily.

We know that the sum of the forces at each node must equal the externally applied force at that point. There are two equations for each point, the horizontal equation and the vertical equation. We can write the equations as:

<table>
<thead>
<tr>
<th></th>
<th>$F_1$</th>
<th>$F_2$</th>
<th>$F_3$</th>
<th>$F_4$</th>
<th>$F_5$</th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_3$</th>
<th>External</th>
</tr>
</thead>
<tbody>
<tr>
<td>1H</td>
<td>$-F_1$</td>
<td></td>
<td></td>
<td></td>
<td>$-F_5$</td>
<td></td>
<td></td>
<td></td>
<td>$-E$</td>
</tr>
<tr>
<td>1V</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>2H</td>
<td>$F_1$</td>
<td></td>
<td>$+.707F_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>2V</td>
<td></td>
<td>$-F_2$</td>
<td>$-.707F_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>3H</td>
<td></td>
<td>$F_2$</td>
<td></td>
<td>$+F_4$</td>
<td></td>
<td>$+R_1$</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>3V</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$+R_3$</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>4H</td>
<td>$-.707F_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>4V</td>
<td>$+.707F_3$</td>
<td></td>
<td></td>
<td>$+F_5$</td>
<td></td>
<td></td>
<td>$+R_2$</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

The row above the equations lists all of the unknowns. The equations are written to keep the unknown terms in their respective columns. The column on the left lists the equations needed to solve the forces in the system. The number in this column refers to the node, the $H$ for horizontal forces, and $V$ for vertical forces. Empty cells in the equation table indicate the force or reaction specified in the column does not apply to that particular node. These empty cells could be filled with zeros.

The coefficients and sign of the coefficients for the element forces can be readily determined by looking at the direction of the element with respect to the node we are analyzing. This is shown in the diagram below.

![Diagram](image)

There are several things to notice about the equations above:

1. There are 8 equations and 8 unknowns. This system of equations can be solved for the forces in the elements and the reactions.

2. In this particular problem, there are 5 elements and one unknown force per element. There are also 3 unknown reactions. This gives us a total of 8
unknowns in the problem. There are 2 equations for each node, a vertical equation and a horizontal equation which gives us a total of 8 equations. We can solve this system since we have as many equations as we have unknowns. In general, problems of this type must satisfy the equation shown below if they are solvable.

\[ 2 \times \text{Nodes} = \text{Elements} + \text{Reactions} \]

3. If we add another element as shown at the right, there will be 8 equations and 9 unknowns. The problem will no longer be statically determinate and cannot be solved using the technique we are discussing.

4. In this type of problem, there must be at least 3 reactions. Two reactions are required to eliminate the X and Y translation and another reaction is required to eliminate any rotation of the object. The basic premise is that the object cannot move. It is in static equilibrium.

### II Matrix Notation

The equations above can be rewritten in matrix notation as shown below.

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & .707 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -.707 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -.707 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & .707 & 0 & 1 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
F_5 \\
R_1 \\
R_2 \\
R_3
\end{bmatrix}
= 
\begin{bmatrix}
E_1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

Or as:

\[
[M] \times \{F\} = \{E\}
\] (2.1)

Or as:

\[
[M] \times \{F\} = \{E\}
\]

Matrix based upon shape of the structure

External Forces

Forces and Reactions
This structure is common to many types of engineering problems. The left-hand-side is completely dependent upon the geometry and the right-hand-side upon the driving forces. If the geometry does not change, we can examine many load cases without changing the left-hand-side of the equation. This can lead to solution efficiencies we will discuss later.

III Matrix Algebra Review

3.0 Matrix Multiplication

Matrix multiplication is a relatively simple operation where the rows of the first matrix are multiplied times the columns of the second matrix. The results are shown below.

\[
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{bmatrix}
\times
\begin{bmatrix}
  j & k & l \\
  m & n & o \\
  p & q & r
\end{bmatrix}
= 
\begin{bmatrix}
  aj + bm + cp & ak + bn + cq & al + bo + cr \\
  dj + em + fp & dk + en + fq & dl + eo + fr \\
  gj + hm + ip & gk + hn + iq & gl + ho + ir
\end{bmatrix}
\]  

(2.3)

In matrix notation we can write:

\[
[A] \times [B] = [C] 
\]

(2.4)

which is an equivalent statement. Note that in general, matrix multiplication is not commutative so that:

\[
[B] \times [A] \neq [C].
\]

(2.5)

You can easily prove this by multiplying matrix [B] times [A]. The product will not be equal to [C]. For the two matrices to be equal, each term of the two matrices must be equal.

3.1 Matrix Transpose

A matrix transpose is created by swapping the rows and columns of a matrix. This is shown below.

\[
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{bmatrix}^T =
\begin{bmatrix}
  a & d & g \\
  b & e & h \\
  c & f & i
\end{bmatrix}
\]

(2.6)

The superscript T indicates the transpose operation.
3.2 Identity Matrix

The identity matrix is a special matrix composed of 1s on the diagonal and 0s everywhere else.

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  

(2.7)

The identity matrix has a property similar to the scalar number 1. Any matrix multiplied by the identity matrix is equal to itself. This is shown below. \([I]\) is the identity matrix and \([M]\) is any other matrix.

\[ [I] \times [M] = [M] \]

(2.8)

or

\[ [M] \times [I] = [M] \]

(2.9)

3.3 Matrix Inverse

If the matrix \([B]\) is the inverse of \([A]\) then:

\[ [B] \times [A] = [I] \]

(2.10)

Another way of stating this is:

\[ [A]^{-1} \times [A] = [I] \]

(2.11)

The -1 superscript indicates the inverse of a matrix.

The inverse of a matrix cannot be easily created with simple row column operations as could the transpose of a matrix but it does have important uses. In solving problems that can be represented mathematically as shown in equation (2.2)

\[ [M] \times \{F\} = \{E\} \]

(2.2)

We can multiply both sides by the inverse of \([M]\)

\[ [M]^{-1} \times [M] \times \{F\} = [M]^{-1} \times \{E\} \]

(2.12)

or
\[
\{F\} = [M]^{-1} \times \{E\} \tag{2.13}
\]

which simplifies solving the equations to a simple matrix – vector multiplication. This can be a useful technique if the forces are needed for many different load cases.

### 3.4 Row and Column Vectors

Frequently you will encounter row and vector columns. They can contain the same information but are written differently. In a row vector, all of the terms are written horizontally and in a column vector, all of the terms are written vertically.

\[
\begin{array}{c}
\text{Column Vector} \\
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
\end{array} \quad \begin{array}{c}
\text{Row Vector} \\
\{a \ b \ c\}
\end{array} \tag{2.14}
\]

You will frequently see a column vector written as \{a \ b \ c\}^T where the transpose of a row vector is a column vector.

### 3.4 Vector – Vector Multiplication

A vector multiplied by another vector results in a scalar. This is shown in the following example.

\[
\{a \ b \ c\} \times \begin{bmatrix}
d \\
e \\
f
\end{bmatrix} = ad + be + cf \tag{2.15}
\]

Notice that we are multiplying a row vector by a column vector. You cannot multiply two row or two column vectors.

### 3.5 Matrix Vector Multiplication

A vector can be multiplied by a matrix. The result is a vector as shown below.

\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix} \times \begin{bmatrix}
j \\
k \\
l
\end{bmatrix} = \begin{bmatrix}
aj + bk + cl \\
dj + ek + fl \\
gj + hk + il
\end{bmatrix} \tag{2.16}
\]
1. In the problem above, how many equations will be generated?

2. In the problem above, how many unknowns are there?

3. In the problem above, each element is 10 feet long. Construct the matrix you would solve to find the forces in the elements and the reactions. Use the element and node numbering shown.

4. In the problem above, solve for the forces in the elements and the reactions. Use the node and element numbering shown. Element 1 is 5 feet long, element 2 is 4 feet long, and element 3 is 3 feet long.