2D Triangular Elements

8.0 Two Dimensional FEA

Frequently, engineers need to compute the stresses and deformation in relatively thin plates or sheets of material and finite element analysis is ideal for this type of computations. We will look at the development of finite element scheme based on triangular elements in this chapter. We will follow basically the same path we used in developing the FEA techniques for spring systems and trusses.

In both cases, we developed an equation for potential energy and used that equation to develop a stiffness matrix. In the development of the truss equations, we started with Hook’s law and developed the equation for potential energy.

\[ F = k \Delta x \quad (6.1) \]

\[ u = k \int_0^Q x dx = \frac{1}{2} kQ^2 \quad (6.2) \]

From here we developed linear algebraic equations describing the displacement of the nodes (end points) on the truss elements to define a stiffness matrix

\[
\begin{bmatrix}
  c^2 & cs & -c^2 & -cs \\
  cs & s^2 & -cs & -s^2 \\
  -c^2 & -cs & e^2 & cs \\
  -cs & -s^2 & cs & s^2
\end{bmatrix}
\]

\[ k = \frac{AE}{L} \quad (6.21) \]

We used this elementary stiffness matrix to create a global stiffness matrix and solve for the nodal displacements using 7.12.

\[ KQ = F \quad (7.12) \]

We are going to use a very similar development to create FEA equations for a two dimensional flat plate.

8.1 Potential Energy

The potential energy of a spring is computed by integrating the force over the displacement of the spring as shown in equation 6.2. We will use the same idea but express it in a slightly different manner since we are not working with a one dimensional object such as a spring.

If we apply forces to a thin plate, the plate will deform and in the process store potential energy much the same way a spring will when an external force is applied. If we look at a small element of material in a plate that has been deformed, we can use the stress, \( \sigma \) to represent the force in the material and the strain, \( \varepsilon \) to represent the displacement of the material. The product of these can be integrated over the volume to
compute the potential energy due to external forces applied to the object. This is shown in the equation 8.1.

\[
U = \frac{1}{2} \int_v \varepsilon^T \sigma dV \quad (8.1)
\]

In 8.1 we are integrating over the entire volume. Since we are studying a flat plate of constant thickness, we can rewrite the equation as

\[
U = \frac{1}{2} \int_A \varepsilon^T \sigma t dA \quad (8.2)
\]

where:
- \(\varepsilon\) is the strain in a differential element of the plate
- \(\sigma\) is the stress in a differential element of the plate
- \(t\) is the thickness of the plate (we assume it is a constant)
- \(A\) is the area of the plate

In this equation, we are expressing the volume as the area of the plate times the thickness of the plate. We will use this equation for potential energy to develop the stiffness matrix for triangular elements in a thin plate. Our goal in this development is to replace both the stress and strain terms with linear equations for nodal displacement.

### 8.2 FEA Elements

We can take a thin plate and divide it into triangles as shown in Figure 1 below.
The triangles share vertices with other triangles. The vertices are nodes and triangles are elements. We will use the elements and nodes to approximate the shape of the object and to compute the displacement of points inside the boundary of the object.

The object is fixed along part of the boundary and does not move. External forces are applied at points. These external forces may arise from simple point forces, tractions or forces applied along a length of the boundary, or body forces such as gravity. Regardless of the source, all forces are applied at the nodes only. Tractions, and body forces may be distributed across several nodes but they are still applied at the nodes.

8.3 Two dimensional Stress – Strain Relationship

Previously we looked at using finite elements to solve for the nodal displacements along a one dimensional truss member. We derived the equation

\[ \sigma = E\varepsilon \]  

(6.22)

Where

- \( \sigma \) is the stress
- \( \varepsilon \) is the strain
- \( E \) is Young’s modulus

For the two dimensional case, this becomes a little more complex. If we look at a two dimensional element, we have

![Figure 2 Element showing both normal and shear stresses](image)

The stresses shown in the figure above can be used to write strain equations.
\[ \varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \]  \hspace{1cm} (8.3)

\[ \varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} \]  \hspace{1cm} (8.4)

\[ \gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy} \]  \hspace{1cm} (8.5)

Where:
- \( \sigma \) is the axial stress
- \( \varepsilon \) is the axial strain
- \( \tau \) is the shear stress
- \( \gamma \) is the shear strain
- \( E \) is Young’s modulus
- \( \nu \) is Poisson’s ratio

We use the equations above to solve for the stress. First we solve 8.4 for \( \sigma_y \), resulting in
\[ \sigma_y = E\varepsilon_y + \nu \sigma_x \]  \hspace{1cm} (8.6)

Substituting this into equation 8.3 yields
\[ \varepsilon_x = \frac{\sigma_x}{E} - \nu \left( E\varepsilon_y + \nu \sigma_x \right) \frac{1}{E} \]  \hspace{1cm} (8.7)

or
\[ E\varepsilon_x = \sigma_x - \nu E\varepsilon_y + \nu^2 \sigma_x \]  \hspace{1cm} (8.8)

Solving for \( \sigma_x \) gives us
\[ \sigma_x = \frac{E}{(1+\nu^2)} \left( \varepsilon_x - \nu \varepsilon_y \right) \]  \hspace{1cm} (8.9)

For the other equations
\[ \sigma_y = \frac{E}{(1+\nu^2)} \left( \varepsilon_y - \nu \varepsilon_x \right) \]  \hspace{1cm} (8.10)

and
\[ \tau_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy} \]  \hspace{1cm} (8.11)

We can write this in vector form as
\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
= \frac{E}{1+\nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]  \hspace{1cm} (8.12)
or
\[ \sigma = D\varepsilon \quad (8.13) \]

where
\[ D = \frac{E}{1 + \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \quad (8.14) \]

At this point, we are about half way to developing the stiffness matrix for the triangular mesh. We can use equation 8.13 to rewrite equation 8.2 so that
\[ U = \frac{1}{2} \int_A \varepsilon^T \sigma \, dA \quad (8.2) \]

becomes
\[ U = \frac{1}{2} \int_A \varepsilon^T D \varepsilon \, dA \quad (8.15) \]

We have eliminated the stress term in the equation. We will go on from here to eliminate the strain term and develop the stiffness matrix.

8.4 2D Triangular Elements

In the two dimensional truss problem, we computed the displacements of the nodes and we will do the same here. We will have displacements in the X and Y directions and we will number them as shown in Figure 3.

![Figure 3 Diagram showing the numbering of nodal displacements.](image)
For a single triangle we have

![Diagram of a triangle showing the numbering of the displacements of its nodes.](image)

We can write the local displacement vectors for each triangle as

\[ \mathbf{q} = \{ q_1, q_2, q_3, q_4, q_5, q_6 \}^T \] (8.16)

For the whole object the global vectors can be written as

\[ \mathbf{Q} = \{ Q_1, Q_2, Q_3, \ldots, Q_n \}^T \] (8.17)

Which includes all of the \( q_n \) terms.

### 8.5 Shape Functions

We are going to compute the displacement of the nodes but we also need to compute the displacement for points inside the triangle. We will use shape functions to interpolate the nodal displacements to compute the displacements of arbitrary points inside the triangles.

We will start by moving only one point on the triangle and holding the other two fixed. We can draw both the deformed and non-deformed triangles on top of one another as shown in Figure 5.
From the diagram above, it is easy to see that points near nodes 2 and 3 will not move as far as points near node 1 when the triangle deforms. We will assume the deformation is linear and we will compute it with areas. The area of a triangle is

\[ \text{Area} = \frac{1}{2} \text{Base} \times \text{Height} \]  

(8.18)

and since we are holding two points fixed and moving the third, the base of the triangle base remains fixed and only the height is changing. This makes the change in area a linear function with its only variable being the height of the triangle or the displacement of the node.

All three nodes of the triangle can be displaced and we will write three linear functions to describe the displacement of an interior point due to the displacement of each of the triangles points. The displacement of the interior node will be computed by summing the displacement due to each three triangle nodes.

The interior point in Figure 6 divides the triangle into 3 regions.

![Figure 5](image1.png)

**Figure 5** Triangle in both non-deformed and deformed states.

All 3 nodal points may move and the motion of the interior point is some combination of their displacement. Let \( A_1 \), \( A_2 \), and \( A_3 \) be the areas of each of triangular regions and \( A \) the total area of the element. We can see from the diagram that
We can derive shape functions

\[
N_1 = \frac{A_1}{A}, \quad N_2 = \frac{A_2}{A}, \quad \text{and} \quad N_3 = \frac{A_3}{A}
\]  

The displacement of the interior point can be computed with the equations 8.21 and 8.22. The displacement \( u \) is in the X direction and \( v \) is in the Y direction.

\[
u = N_i q_i + N_2 q_3 + N_3 q_5
\]
\[
u = N_i q_2 + N_2 q_4 + N_3 q_6
\]

The shape functions are not independent of one another because:

\[
N_1 + N_2 + N_3 = 1
\]

Knowing two of the shape functions makes it possible to compute the third. Because of this we can let

\[
N_1 = \xi, \quad N_2 = \eta, \quad \text{and} \quad N_3 = 1 - \xi - \eta
\]

Substituting these equations into 8.21 and 8.22 yields

\[
u = (q_1 - q_5)\xi + (q_3 - q_5)\eta + q_5
\]
\[
u = (q_2 - q_6)\xi + (q_4 - q_6)\eta + q_6
\]

We can use these same shape functions to compute the coordinates of a point interior to the triangle.

\[
x = N_1 x_1 + N_2 x_2 + N_3 x_3
\]
\[
y = N_1 y_1 + N_2 y_2 + N_3 y_3
\]

Making the substitution to \( \xi \) and \( \eta \) gives us

\[
x = (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3
\]

and

\[
y = (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3
\]
These equations can be used to compute the shape functions. Given some point in the triangle (See Figure 7). We know $x_1, y_1, x_2, y_2, x_3, y_3$, and $x, y$ so we solve equations 8.29 and 8.30 for $\xi$ and $\eta$. If we know the displacements at the nodes, we can use the same shape functions to compute the displacement for the point at $x, y$.

**8.6 Elementary Solid Mechanics**

If we have a small element of material

![Diagram of displacement in a small element of material](image)

Figure 8 Displacement in a small element of material.

And $u$ and $v$ are the displacements across the element, then we can write the strain as

$$\varepsilon = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Strain in the X direction

Strain in the Y direction

Shear strain

(8.31)
We have equations for $u$ and $v$ but these equations are expressed in terms of $\xi$ and $\eta$ not $x$ and $y$. But, using the chain rule

\[
\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \xi} \quad (8.32)
\]

\[
\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \eta} \quad (8.33)
\]

We can write this in matrix form as

\[
\begin{bmatrix}
\frac{\partial u}{\partial \xi} \\
\frac{\partial u}{\partial \eta}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y}
\end{bmatrix} \quad (8.34)
\]

or

\[
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{bmatrix}^{-1}
\begin{bmatrix}
\frac{\partial u}{\partial \xi} \\
\frac{\partial u}{\partial \eta}
\end{bmatrix} \quad (8.35)
\]

We can use equations 8.29 and 8.30 to compute the derivatives in the matrix above.

\[
\frac{\partial x}{\partial \xi} = x_1 - x_3 \quad (8.36)
\]

\[
\frac{\partial x}{\partial \eta} = x_2 - x_3 \quad (8.37)
\]

\[
\frac{\partial y}{\partial \xi} = y_1 - y_3 \quad (8.38)
\]

\[
\frac{\partial y}{\partial \eta} = y_2 - y_3 \quad (8.39)
\]

We can simplify the equations somewhat by letting

\[
x_{ij} = x_i - x_j \quad (8.40)
\]

and

\[
y_{ij} = y_i - y_j \quad (8.41)
\]

Substituting equations 8.36 through 8.41 into equation 8.35 yields
\[
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y}
\end{bmatrix} = \begin{bmatrix}
x_{13} & y'_{13} \\
x_{23} & y'_{23}
\end{bmatrix}^{-1} \begin{bmatrix}
\frac{\partial u}{\partial \xi} \\
\frac{\partial u}{\partial \eta}
\end{bmatrix}
\]  
(8.42)

From linear algebra we know that if

\[
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\]  
(8.43)

Then

\[
A^{-1} = \frac{1}{\det A} \begin{bmatrix}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{bmatrix}
\]  
(8.44)

We also know that the Jacobian of a matrix is defined as

\[
J = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \\
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}
\end{bmatrix}
\]  
(8.45)

and incidentally, the area of the triangle can be defined as

\[
Area = \frac{1}{2} |\det J|
\]  
(8.46)

Putting this together gives us

\[
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y}
\end{bmatrix} = \frac{1}{\det J} \begin{bmatrix}
y'_{23} & -y'_{13} \\
x'_{23} & -x'_{13}
\end{bmatrix} \begin{bmatrix}
\frac{\partial u}{\partial \xi} \\
\frac{\partial u}{\partial \eta}
\end{bmatrix}
\]  
(8.47)

Or by multiplying through

\[
\frac{\partial u}{\partial x} = \frac{1}{\det J} \left(y'_{23} \frac{\partial u}{\partial \xi} - y'_{13} \frac{\partial u}{\partial \eta}\right)
\]  
(8.48)

\[
\frac{\partial u}{\partial y} = \frac{1}{\det J} \left(-x'_{23} \frac{\partial u}{\partial \xi} + x'_{13} \frac{\partial u}{\partial \eta}\right)
\]  
(8.49)
We can now use equation 8.25 to compute the remaining derivatives on the right hand side of equation 8.47

$$u = (q_1 - q_5) \xi + (q_3 - q_5) \eta + q_5 \quad (8.25)$$

so

$$\frac{\partial u}{\partial x} = \frac{1}{\det J} \left( y'_{23} (q_1 - q_5) - y_{13} (q_3 - q_5) \right) \quad (8.50)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\det J} \left( -x'_{23} (q_1 - q_5) - x_{13} (q_3 - q_5) \right) \quad (8.51)$$

Using a similar process for \(v\) we find that

$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} y'_{23} \frac{\partial v}{\partial \xi} - y_{13} \frac{\partial v}{\partial \eta} \\ -x'_{23} \frac{\partial v}{\partial \xi} + x_{13} \frac{\partial v}{\partial \eta} \end{bmatrix} \quad (8.52)$$

From equation 8.26 we can again compute the derivatives on the right hand side of equation 8.52

$$v = (q_2 - q_6) \xi + (q_4 - q_6) \eta + q_6 \quad (8.26)$$

Resulting in

$$\frac{\partial v}{\partial x} = \frac{1}{\det J} \left( y'_{23} (q_2 - q_6) - y_{13} (q_4 - q_6) \right) \quad (8.53)$$

$$\frac{\partial v}{\partial x} = \frac{1}{\det J} \left( -x'_{23} (q_2 - q_6) - x_{13} (q_4 - q_6) \right) \quad (8.54)$$

So the strain defined in equation 8.31

$$\mathbf{\varepsilon} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} \quad (8.31)$$
Becomes

\[
\varepsilon = \frac{1}{\det J} \begin{bmatrix} 
 y_{23} (q_1 - q_5) - y_{13} (q_3 - q_5) \\
-x_{23} (q_2 - q_6) + x_{13} (q_4 - q_6) \\
-x_{23} (q_1 - q_5) + x_{13} (q_3 - q_5) + y_{23} (q_2 - q_6) - y_{13} (q_4 - q_6) 
\end{bmatrix}
\]  

(8.55)

We can simplify this equation by combining terms. There are many relationships we can make using the x and y terms by realizing that

\[ y_{12} = y_1 - y_2 \]  

(8.56)

Now adding and subtracting \( y_3 \) from the right hand side

\[ y_{12} = y_1 - y_3 - y_2 + y_3 \]  

(8.57)

so

\[ y_{12} = y_{13} - y_{23} \]  

(8.58)

Using this type of substitution allows us to rewrite equation 8.55 as

\[
\varepsilon = \frac{1}{\det J} \begin{bmatrix} 
 y_{23} q_1 + y_{31} q_3 + y_{12} q_5 \\
x_{23} q_2 + x_{13} q_4 + x_{23} q_6 \\
x_{23} q_1 + y_{23} q_2 + x_{13} q_3 + y_{13} q_4 + x_{21} q_5 + y_{12} q_6 
\end{bmatrix}
\]  

(8.59)

Writing the equation in matrix form we get

\[ \varepsilon = Bq \]  

(8.60)

Where \( B \) is a 3x6 element strain displacement matrix relating 3 strains to 6 nodal displacements.

\[
B = \frac{1}{\det J} \begin{bmatrix} 
 y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\
0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\
x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} 
\end{bmatrix}
\]  

(8.61)

Substituting Equation 8.60 into our potential energy equation 8.14

\[ U = \frac{1}{2} \int_A \varepsilon^T \det J dA \]  

(8.15)
gives us

\[ U = \frac{1}{2} \int_A q^T B^T DBqdA \]  
(8.62)

If we are looking at a single triangle we can rewrite this equation as

\[ U_e = \frac{1}{2} \int_e q^T B^T DBqt_e dA \]  
(8.63)

The thickness of the plate \( t \) is a constant as are the matrices \( B \) and \( D \). We can move these outside the integral resulting in

\[ U_e = \frac{1}{2} q^T B^T DBt_e \int_e dA \]  
(8.64)

We recognize the integral \( \int_e dA \) as just the area of the triangle so our equation becomes

\[ U_e = \frac{1}{2} q^T t_e A_e B^T DBq \]  
(8.65)

We can now represent the stiff matrix for the triangle as

\[ k_e = t_e A_e B^T DB \]  
(8.66)

With this substitution, our potential energy \( U \) becomes

\[ U_e = \frac{1}{2} q^T k_e q \]  
(8.67)

We can sum these individual triangles to compute the strain energy over the entire plate giving us the equation

\[ U = \sum_e \frac{1}{2} q^T k_e q \]  
(8.68)

or

\[ U = \frac{1}{2} Q^T KQ \]  
(8.69)
In this equation $Q$ is the global displacement vector which is the sum of all the local displacement vectors and $K$ is the global stiffness matrix which is the sum of all the local stiffness matrices.

We now have what we need to solve for the displacements in our familiar equation

$$KQ = F \quad (8.70)$$